Three-loop planar master integrals for heavy-to-light form factors

arXiv:1810.04328

陈龙斌(Long-Bin Chen)

广州大学

2019.1.24

The need of higher order QCD corrections

QED $\alpha \sim O(0.01)$

QCD $\alpha_s \sim O(0.1)\ u \gg \Lambda_{QCD}$

QED are more convergence
in perturbation expansion than **QCD**

Top quark

1. The heaviest particle in standard model

2. Suitable for QCD perturbative calculations

3. Huge samples of top quarks at the LHC

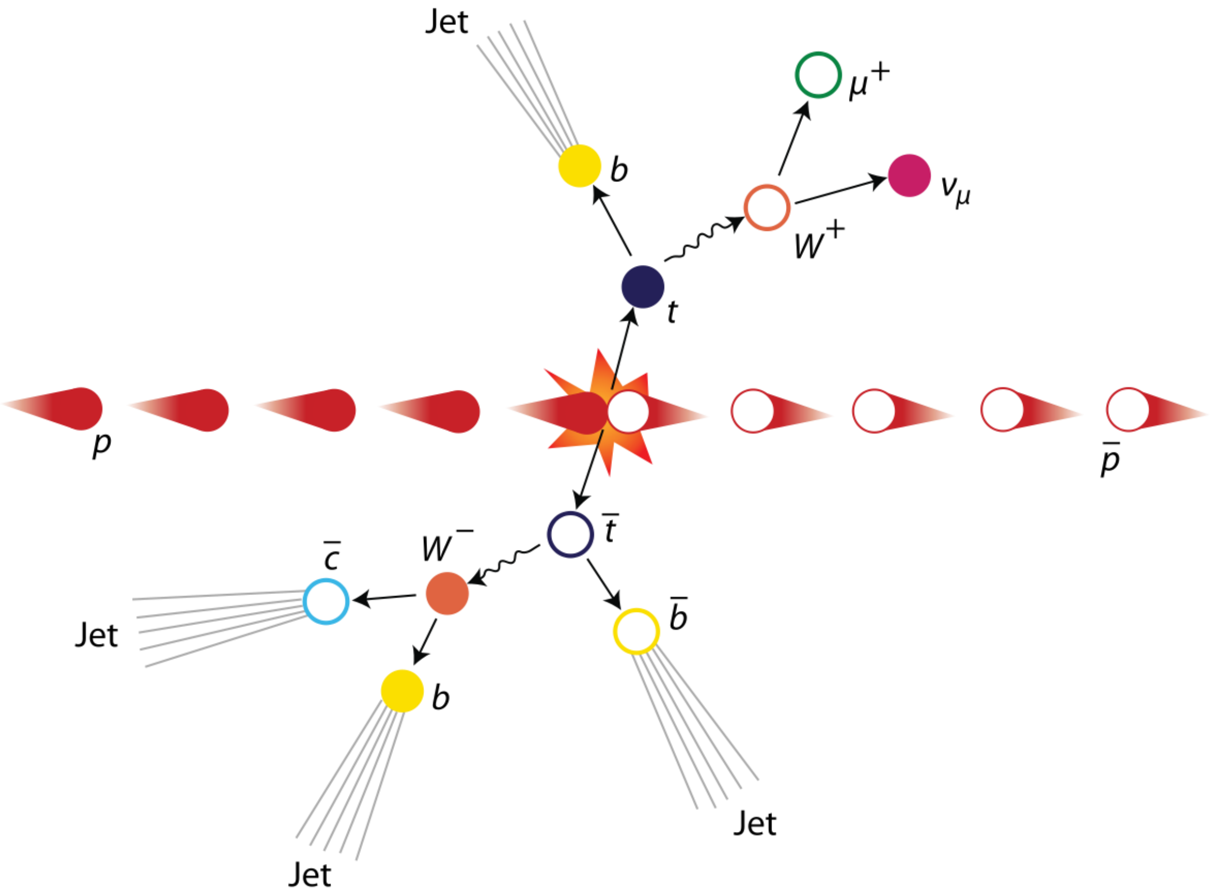
4. Study of CKM matrix element

5. Search for new physics beyond standard model

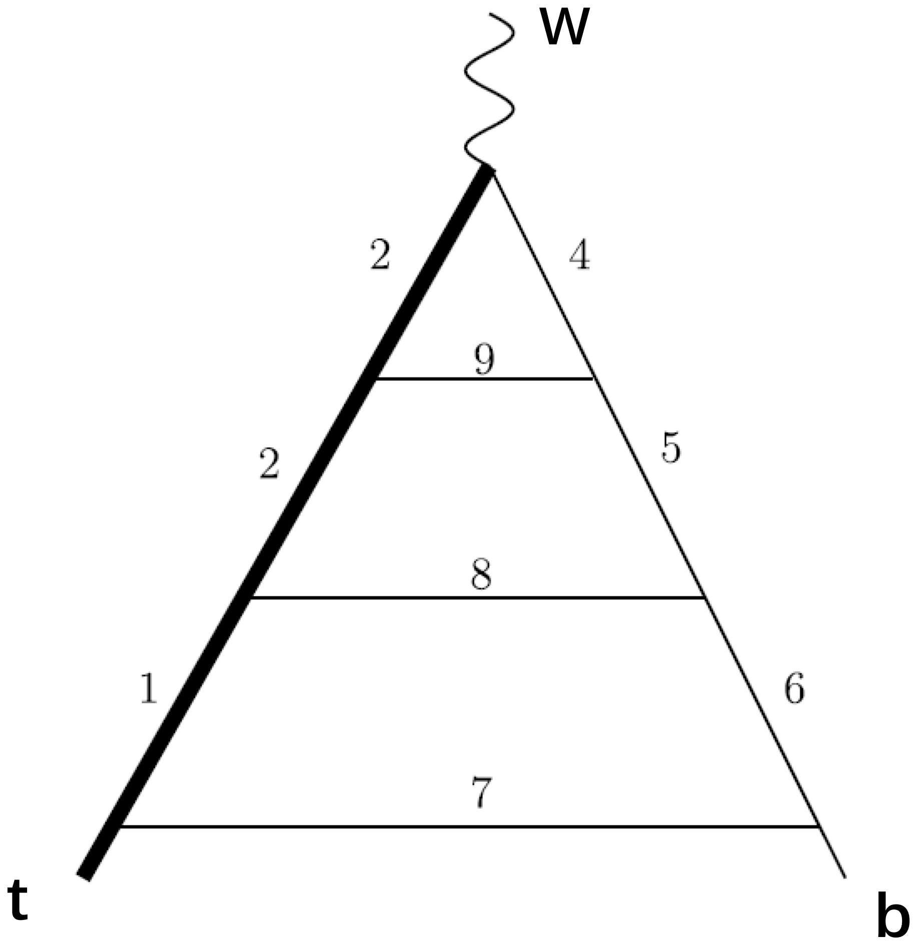
…………………………….

Top Quark Decay at Next-to-Next-to Leading Order in QCD
Jun Gao (Southern Methodist U.), Chong Sheng Li (Peking U. & Peking U., CAPT & Peking U.), Hua Xing Zhu (SLAC). Oct 2012. 5 pp.
Published in **Phys.Rev.Lett. 110 (2013) no.4, 042001**

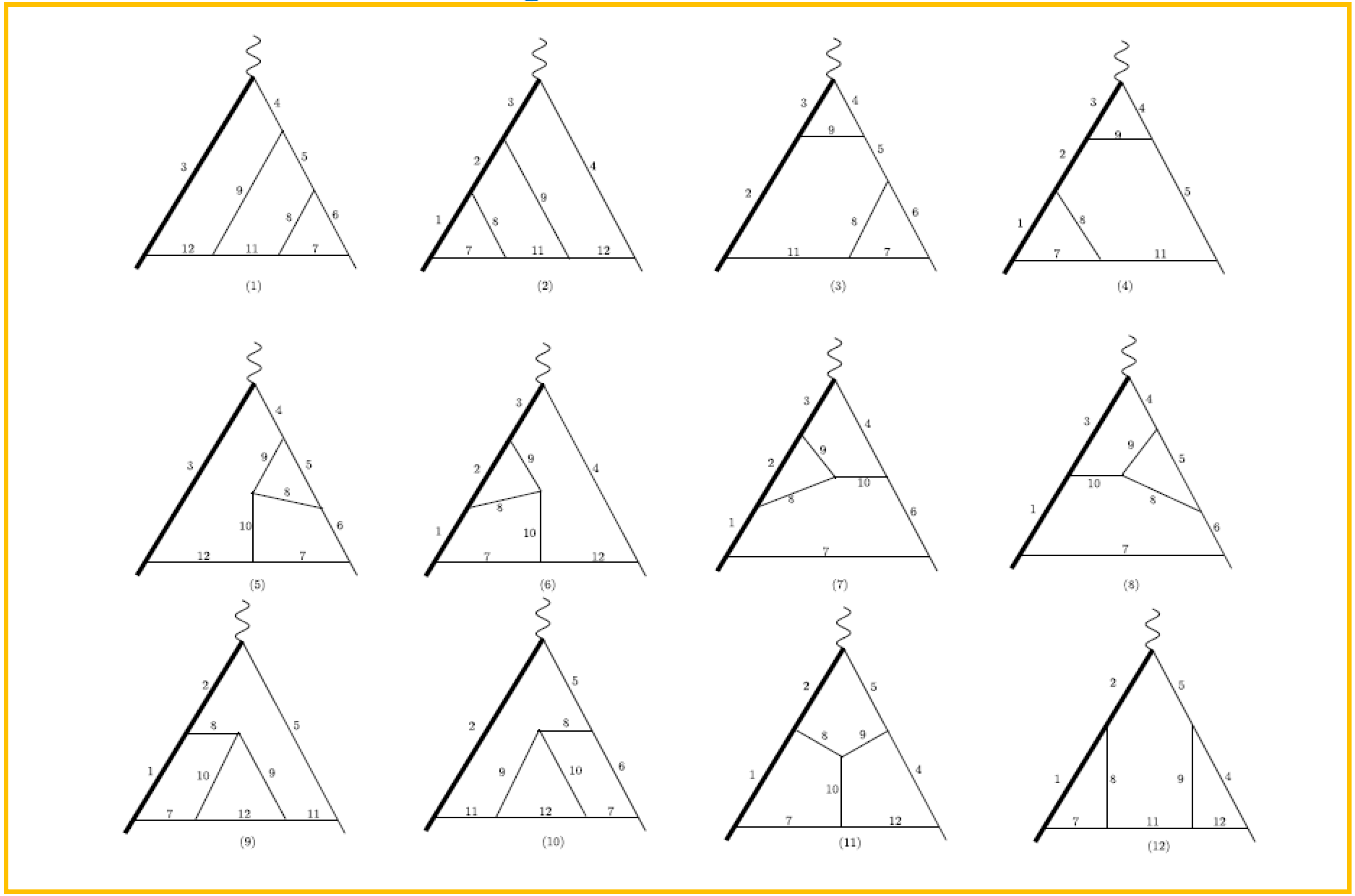
Charm-Quark Production in Deep-Inelastic Neutrino Scattering at Next-to-Next-to-Leading Order in QCD
Edmond L. Berger, Jun Gao (Argonne), Chong Sheng Li (Peking U. & Peking U., CHEP & Peking U., SKLNPT), Ze Long Liu (Peking U. & Peking U., SKLNPT), Hua Xing Zhu (MIT, Cambridge, CTP). Jan 20, 2016. 6 pp.
Published in **Phys.Rev.Lett. 116 (2016) no.21, 212002**


Jet
b
μ+
νμ
W+
t
p
p̄
t̄
W−
c̄
Jet
b
b̄
Jet
Jet

Heavy-to-light Form factors



Color-planar diagrams (Leading Color Contribution)



Integrals can be parameterized by

d=4-2∈

$$I_{n_1,n_2,\ldots,n_{12}} = \int \frac{\mathcal{D}^d k_1\, \mathcal{D}^d k_2\, \mathcal{D}^d k_3}{D_1^{n_1}\, D_2^{n_2}\, D_3^{n_3}\, D_4^{n_4}\, D_5^{n_5}\, D_6^{n_6}\, D_7^{n_7}\, D_8^{n_8}\, D_9^{n_9}\, D_{10}^{n_{10}}\, D_{11}^{n_{11}}\, D_{12}^{n_{12}}}$$

$$D_1 = -(k_1+p_1)^2 + m^2,\; D_2 = -(k_2+p_1)^2 + m^2,$$

$$D_3 = -(k_3+p_1)^2 + m^2,\; D_4 = -(k_3+p_2)^2,$$

$$D_5 = -(k_2+p_2)^2,\; D_6 = -(k_1+p_2)^2,\; D_7 = -k_1^2,$$

$$D_8 = -(k_1-k_2)^2,\; D_9 = -(k_2-k_3)^2,\; D_{10} = -(k_1-k_3)^2,$$

$$D_{11} = -k_2^2,\; D_{12} = -k_3^2,$$

All integrals can be reduced to 71 Master Integrals (MI)

Integration-By-Parts (IBP) reduction

(FIRE, Reduze, Kira···)

Example

$$F(a_1, a_2) = \int \frac{\mathrm{d}^d k}{(k^2)^{a_1}[(q-k)^2]^{a_2}}.$$

$$\int \mathrm{d}^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2)^{a_1}[(q-k)^2]^{a_2}} = 0$$

$$F(a_1, a_2) = -\frac{1}{(a_2-1)q^2}[(d-2a_1-a_2+1)F(a_1, a_2-1) - (a_2-1)F(a_1-1, a_2)].$$

Calculations of Master Integrals

- Evaluating by Feynman Parameters
- Evaluating by Mellin-Barnes Integrals

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz\, \Gamma(\lambda+z)\Gamma(-z) \frac{Y^{z}}{X^{\lambda+z}} .$$

- Sector Decompositions (Numeric Calculations)
- ....

Differential Equations (DE)

$$\frac{d}{dx}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{n,n} \end{pmatrix}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

x are Lorentz invariant kinematics

A. V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett.* **B254** (1991) 158–164.

A. V. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, *Phys. Lett.* **B267** (1991) 123–127. [Erratum: Phys. Lett.B295,409(1992)].

A suitable choice of basis (canonical basis)
arXiv:1304.1806

Multiloop integrals in dimensional regularization made simple
Johannes M. Henn (Princeton, Inst. Advanced Study). Apr 5, 2013. 4 pp.
Published in **Phys.Rev.Lett. 110 (2013) 251601**
DOI: 10.1103/PhysRevLett.110.251601
e-Print: **arXiv:1304.1806 [hep-th] | PDF**

$$\partial_x \vec{g}(x;\epsilon) = B(x,\epsilon)\,\vec{g}(x;\epsilon)$$

$$\vec{f} = T\vec{g},$$

$$B = T^{-1}AT - T^{-1}\partial_x T$$

$$d\,\vec{f}(x,\epsilon) = \epsilon\,\left(d\,\tilde{A}\right)\,\vec{f}(x;\epsilon)$$

$$\tilde{A} = \left[\sum_k A_k \log \alpha_k(x)\right]\,.$$

$$\frac{\partial}{\partial s} = \frac{1}{s-m^2} p_2 \cdot \frac{\partial}{\partial p_2}$$

Canonical Basis:

$$
\begin{aligned}
F_1 &= m^6\, I_{3,3,3,0,0,0,0,0,0,0,0,0}\,, \\
F_2 &= \epsilon^2 m^4\, I_{0,2,3,0,0,0,1,2,0,0,0,0}\,, \\
F_3 &= \epsilon^3 m^2\, I_{0,0,2,0,0,0,2,2,1,0,0,0}\,, \\
F_4 &= (\epsilon-1)(1+4\epsilon)\epsilon\, m^2\, I_{2,0,2,0,0,0,0,2,1,0,0,0}\,, \\
F_5 &= \epsilon\, s\, m^4\, I_{3,3,2,1,0,0,0,0,0,0,0,0}\,, \\
F_6 &= \epsilon^3 s\, I_{2,0,0,2,0,0,0,2,1,0,0,0}\,, \\
F_7 &= \epsilon^2 m^2\, (2\epsilon\, I_{2,0,0,2,0,0,0,2,1,0,0,0} \\
&\quad + (s-m^2)\, I_{3,0,0,2,0,0,0,2,1,0,0,0})\,,
\end{aligned}
$$

……………………………………………

$$F_{67} = \epsilon^5\,(s-m^2)\,I_{1,1,0,1,1,-1,1,1,0,2,0,0}\,,$$
$$F_{68} = \epsilon^5\,(s-m^2)^2\,I_{1,1,0,1,1,0,1,1,0,2,0,0}\,,$$
$$F_{69} = \epsilon^6\,(s-m^2)\,I_{1,1,0,1,0,0,1,1,1,0,0,1}\,,$$
$$F_{70} = \epsilon^6\,(s-m^2)^2\,I_{1,1,0,1,1,0,1,1,1,1,-1,1}\,,$$
$$F_{71} = \epsilon^6\,(s-m^2)\,I_{1,1,0,1,1,-1,1,1,1,1,-1,1} + \frac{1}{12(1-2\epsilon)}(12F_2 + 6F_3 + 3F_4 - 2F_7 + 6F_9 - 18F_{14} + 2F_{24} + 12F_{25})\,.$$

$$\frac{\partial \mathbf{F}(x,\epsilon)}{\partial x} = \epsilon \left(\frac{\mathbf{P}}{x} + \frac{\mathbf{Q}}{x-1} \right) \mathbf{F}(x,\epsilon).$$

P and Q are 71*71 rational matrices

DE for non-canonical basis

$$
\frac{1}{s-m^2}\left(\frac{(2d-7)(5d-18)\left(7m^2+5s\right)G(1,\{1,0,0,0,1,0,0,1,1,1,0,1\})(d-4)^2}{(d-5)(d-3)\left(m^2-s\right)s\left(19dm^2-72m^2-3ds+12s\right)}+\right.
$$

$$
\frac{G(1,\{1,0,1,1,1,0,1,1,1,0,0,0\})(d-4)^2}{2(d-3)s}+\frac{6(d-6)m^2\left(9dm^2-33m^2+23ds-87s\right)G(1,\{0,1,0,1,0,0,1,1,1,2,0,0\})(d-4)}{(d-5)(d-3)\left(m^2-s\right)s\left(19dm^2-72m^2-3ds+12s\right)}-
$$

$$
\left((2d-7)\left(88d^2m^4-663dm^4+1242m^4+9d^2sm^2-73dsm^2+150sm^2-d^2s^2+16ds^2-48s^2\right)\right.
$$

$$
\left.G(1,\{1,0,1,0,1,0,1,1,1,0,0,0\})(d-4)\right)\Big/\left(3(d-6)(d-3)\left(m^2-s\right)^2s\left(dm^2-3m^2-2ds+7s\right)\right)+
$$

$$
\left(\left(167d^3m^6-1858d^2m^6+6864dm^6-8424m^6+111d^3sm^4-1256d^2sm^4+4762dsm^4-6036sm^4+\right.\right.
$$

$$
\left.232d^3s^2m^2-2563d^2s^2m^2+9388ds^2m^2-11412s^2m^2-30d^3s^3+349d^2s^3-1334ds^3+1680s^3\right)
$$

$$
\left.G(1,\{1,0,1,0,1,0,1,1,2,0,0,0\})(d-4)\right)\Big/\left(3(d-6)(d-3)(3d-10)\left(m^2-s\right)^2s\left(dm^2-3m^2-2ds+7s\right)\right)+
$$

$$
\frac{3\left(5dm^4-18m^4+22dsm^2-84sm^2+5ds^2-18s^2\right)G(1,\{1,1,0,1,0,0,1,1,2,0,0,0\})(d-4)}{2(d-6)(d-3)\left(m^2-s\right)^2s}-
$$

$$
\frac{\left(dm^2-6m^2+5ds-24s\right)G(1,\{1,1,0,1,1,0,1,0,1,2,0,0\})(d-4)}{(d-6)(d-3)s}-
$$

$$
\frac{3G(1,\{1,1,0,1,1,0,1,1,0,1,0,0\})(d-4)}{2s}+\frac{5\left(m^2+s\right)G(1,\{1,1,0,1,1,0,1,1,0,2,0,0\})(d-4)}{2(d-3)s}+
$$

$$
\left(2(d-3)^2\left(85d^3m^6-879d^2m^6+3074dm^6-3648m^6-1149d^3sm^4+13999d^2sm^4-56198dsm^4+74472sm^4-517d^3s^2m^2+\right.\right.
$$

$$
\left.\left.6775d^2s^2m^2-28786ds^2m^2+39936s^2m^2+45d^3s^3-567d^2s^3+2358ds^3-3240s^3\right)G(1,\{0,1,0,0,0,1,0,1,1,0,0,1\})\right)\Big/
$$

$$
\left((d-6)(d-5)(5d-18)m^2\left(m^2-s\right)^3s\left(19dm^2-72m^2-3ds+12s\right)\right)-
$$

$$
\left(\left(85d^4m^8-1219d^3m^8+6590d^2m^8-15944dm^8+14592m^8-6024d^4sm^6+96352d^3sm^6-572876d^2sm^6+1502664dsm^6-\right.\right.
$$

$$
1468704sm^6-13186d^4s^2m^4+214990d^3s^2m^4-1299328d^2s^2m^4+3456840ds^2m^4-3421152s^2m^4+648d^4s^3m^2-
$$

$$
\left.9856d^3s^3m^2+55932d^2s^3m^2-140360ds^3m^2+131424s^3m^2+45d^4s^4-747d^3s^4+4626d^2s^4-12672ds^4+12960s^4\right)
$$

$$
\left.G(1,\{0,1,0,1,0,0,1,1,2,0,0,0\})\right)\Big/\left(2(d-6)(d-5)(5d-18)m^2\left(m^2-s\right)^3s\left(19dm^2-72m^2-3ds+12s\right)\right)-
$$

$$G(1, \{0, 1, 0, 1, 0, 0, 1, 1, 2, 0, 0, 0\})) \big/ \left(2\,(d-6)\,(d-5)\,(5\,d-18)\,m^2\,(m^2-s)^3\,s\,(19\,d\,m^2-72\,m^2-3\,d\,s+12\,s)\right) -$$

$$\big(2\,(15\,136\,d^6\,m^8 - 351\,926\,d^5\,m^8 + 3\,397\,098\,d^4\,m^8 - 17\,429\,796\,d^3\,m^8 + 50\,143\,848\,d^2\,m^8 - 76\,708\,008\,d\,m^8 + 48\,755\,520\,m^8 -$$
$$27\,710\,d^6\,s\,m^6 + 617\,510\,d^5\,s\,m^6 - 5\,702\,807\,d^4\,s\,m^6 + 27\,953\,891\,d^3\,s\,m^6 - 76\,744\,006\,d^2\,s\,m^6 + 111\,931\,368\,d\,s\,m^6 -$$
$$67\,780\,800\,s\,m^6 + 11\,804\,d^6\,s^2\,m^4 - 259\,522\,d^5\,s^2\,m^4 + 2\,380\,136\,d^4\,s^2\,m^4 - 11\,644\,177\,d^3\,s^2\,m^4 + 32\,021\,430\,d^2\,s^2\,m^4 -$$
$$46\,895\,544\,d\,s^2\,m^4 + 28\,553\,760\,s^2\,m^4 + 44\,978\,d^6\,s^3\,m^2 - 1\,032\,122\,d^5\,s^3\,m^2 + 9\,785\,211\,d^4\,s^3\,m^2 - 49\,119\,463\,d^3\,s^3\,m^2 +$$
$$137\,821\,510\,d^2\,s^3\,m^2 - 205\,099\,608\,d\,s^3\,m^2 + 126\,547\,200\,s^3\,m^2 - 2688\,d^6\,s^4 + 59\,496\,d^5\,s^4 - 546\,258\,d^4\,s^4 +$$
$$2\,663\,577\,d^3\,s^4 - 7\,276\,014\,d^2\,s^4 + 10\,558\,992\,d\,s^4 - 6\,360\,480\,s^4)\,G(1, \{1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 2\})\big) \big/$$
$$\left(3\,(d-6)\,(d-5)\,(d-3)\,(3\,d-10)\,(m^2-s)^3\,s\,(19\,d\,m^2-72\,m^2-3\,d\,s+12\,s)\,(d\,m^2-3\,m^2-2\,d\,s+7\,s)\right) +$$

$$\big((633\,d^4\,m^6 - 9198\,d^3\,m^6 + 50\,083\,d^2\,m^6 - 121\,116\,d\,m^6 + 109\,764\,m^6 + 816\,d^4\,s\,m^4 - 12\,011\,d^3\,s\,m^4 + 66\,179\,d^2\,s\,m^4 -$$
$$161\,814\,d\,s\,m^4 + 148\,176\,s\,m^4 + 249\,d^4\,s^2\,m^2 - 3600\,d^3\,s^2\,m^2 + 19\,597\,d^2\,s^2\,m^2 - 47\,580\,d\,s^2\,m^2 +$$
$$43\,452\,s^2\,m^2 + 30\,d^4\,s^3 - 439\,d^3\,s^3 + 2381\,d^2\,s^3 - 5682\,d\,s^3 + 5040\,s^3)\,G(1, \{1, 0, 1, 0, 1, 0, 1, 2, 1, 0, 0, 0\})\big) \big/$$
$$\left(3\,(d-6)\,(d-3)\,(3\,d-10)\,(m^2-s)^2\,s\,(d\,m^2-3\,m^2-2\,d\,s+7\,s)\right) - 1 \big/ \left(2\,(d-6)\,(d-3)\,(m^2-s)^2\,s\,(m^2+s)\right)$$
$$(3\,d^2\,m^6 - 30\,d\,m^6 + 72\,m^6 - 4\,d^2\,s\,m^4 + 50\,d\,s\,m^4 - 132\,s\,m^4 + 87\,d^2\,s^2\,m^2 - 676\,d\,s^2\,m^2 + 1308\,s^2\,m^2 + 10\,d^2\,s^3 - 88\,d\,s^3 + 192\,s^3)$$
$$G(1, \{1, 0, 1, 1, 0, 0, 1, 1, 2, 0, 0, 0\}) - \frac{3\,(3\,d-10)^2\,(m^2+s)\,G(1, \{1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0\})}{2\,(d-6)\,(m^2-s)^2\,s} -$$

$$\big(4\,(d-3)^2\,(2\,d-7)\,(3\,d-10)\,(34\,d^2\,m^6 - 294\,d\,m^6 + 624\,m^6 + 61\,d^2\,s\,m^4 - 421\,d\,s\,m^4 + 720\,s\,m^4 - 156\,d^2\,s^2\,m^2 +$$
$$1184\,d\,s^2\,m^2 - 2244\,s^2\,m^2 - 3\,d^2\,s^3 + 27\,d\,s^3 - 60\,s^3)\,G(1, \{0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0\})\big) \big/$$
$$\left((d-5)\,(5\,d-18)\,(5\,d-16)\,m^2\,(m^2-s)^3\,s\,(19\,d\,m^2-72\,m^2-3\,d\,s+12\,s)\,(d-4)\right) +$$

$$\big((d-3)\,(3\,d^3\,m^4 - 38\,d^2\,m^4 + 157\,d\,m^4 - 210\,m^4 + 20\,d^3\,s\,m^2 - 324\,d^2\,s\,m^2 + 1648\,d\,s\,m^2 - 2688\,s\,m^2 + d^3\,s^2 - 30\,d^2\,s^2 + 211\,d\,s^2 - 430\,s^2)$$
$$G(1, \{0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0\})\big) \big/ \left(2\,(d-5)\,(3\,d-14)\,m^2\,(m^2-s)^3\,s\,(d-4)\right) -$$

$$\big((3\,d^4\,m^6 - 50\,d^3\,m^6 + 309\,d^2\,m^6 - 838\,d\,m^6 + 840\,m^6 + 79\,d^4\,s\,m^4 - 1466\,d^3\,s\,m^4 + 9925\,d^2\,s\,m^4 - 29\,250\,d\,s\,m^4 + 31\,808\,s\,m^4 +$$
$$61\,d^4\,s^2\,m^2 - 1330\,d^3\,s^2\,m^2 + 10\,155\,d^2\,s^2\,m^2 - 32\,958\,d\,s^2\,m^2 + 38\,848\,s^2\,m^2 + d^4\,s^3 - 34\,d^3\,s^3 + 331\,d^2\,s^3 - 1274\,d\,s^3 + 1720\,s^3)$$
$$G(1, \{0, 1, 0, 1, 0, 1, 0, 1, 2, 0, 0, 0\})\big) \big/ \left(4\,(d-5)\,(d-3)\,(3\,d-14)\,m^2\,(m^2-s)^3\,s\,(d-4)\right) -$$

...........(about 30pages).

Boundary Conditions

Known

$$F_1 = \frac{1}{8},$$

$$F_2 = \frac{1}{8} + \epsilon^2 \frac{\pi^2}{12} + \epsilon^3 \zeta(3) + \epsilon^4 \frac{4\pi^4}{45} + 2\epsilon^5 \frac{27\zeta(5) + \pi^2\zeta(3)}{3} + \epsilon^6 \left(\frac{229\pi^6}{1890} + 4\zeta^2(3) \right) + \mathcal{O}(\epsilon^7),$$

$$x \equiv \frac{s}{m^2}.$$

$$\frac{\partial F_{38}}{\partial x} = \epsilon \left(\frac{-4F_2 - F_3 + 6F_{11} + 2F_{19} - 6(3F_{38} - 2F_{39})}{6x} + \frac{2F_{38}}{x-1} \right),$$

Regular at x=0

$$-6(3F_{38} - 2F_{39})|_{x=0} = (-4F_2 - F_3 + 6F_{11} + 2F_{19})|_{x=0}$$

$$
\begin{aligned}
F_{38} &= \epsilon^5 \, (s-m^2) \, I_{0,1,1,1,0,0,1,2,1,0,0,0} \,, \\
F_{39} &= \epsilon^4 \, m^2 \, (s-m^2) \, I_{0,1,2,1,0,0,1,2,1,0,0,0} \,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F_{38}}{\partial x} &= \epsilon \left(\frac{-4F_2 - F_3 + 6F_{11} + 2F_{19} - 6(3F_{38} - 2F_{39})}{6x} + \frac{2F_{38}}{x-1} \right) , \\
\frac{\partial F_{39}}{\partial x} &= \epsilon \left(\frac{-20F_2 + 4F_3 + 3F_{11} - 12F_{16} + 30F_{18} + 16F_{19} - 30(3F_{38} - 2F_{39})}{12x} \right. \\
&\quad \left. -2\frac{3F_{39} - 4F_{38}}{x-1} \right) .
\end{aligned}
$$

$$
\begin{aligned}
-6(3F_{38} - 2F_{39})|_{x=0} &= (-4F_2 - F_3 + 6F_{11} + 2F_{19})|_{x=0} \,, \\
-30(3F_{38} - 2F_{39})|_{x=0} &= (-20F_2 + 4F_3 + 3F_{11} - 12F_{16} + 30F_{18} + 16F_{19})|_{x=0} \,.
\end{aligned}
$$

$$
\begin{aligned}
F_{37}|_{x=0} = &-\frac{11}{240} - \epsilon^2 \frac{\pi^2}{180} + \epsilon^3 \frac{3\zeta(3)}{4} + \epsilon^4 \frac{379\pi^4}{5400} \\
&- \epsilon^5 \left(\frac{4\pi^2\zeta(3)}{3} - \frac{1107\zeta(5)}{20} \right) \\
&+ \epsilon^6 \left(\frac{901\pi^6}{4725} + \frac{21\zeta^2(3)}{2} \right) + \mathcal{O}(\epsilon^7)\,, \\
F_{39}|_{x=0} = &-\epsilon^2 \frac{\pi^2}{36} + \epsilon^4 \frac{79\pi^4}{1080} + \epsilon^5 \left(\frac{143\pi^2\zeta(3)}{18} + \frac{5\zeta(5)}{2} \right) \\
&+ \epsilon^6 \left(\frac{18737\pi^6}{22680} + 48\zeta^2(3) \right) + \mathcal{O}(\epsilon^7)\,, \\
F_{41}|_{x=0} = &\frac{7}{180} - \epsilon^2 \frac{7\pi^2}{270} - \epsilon^3 \frac{89\zeta(3)}{45} - \epsilon^4 \frac{139\pi^4}{900} \\
&- \epsilon^5 \frac{353\pi^2\zeta(3) + 8469\zeta(5)}{135} \\
&- \epsilon^6 \left(\frac{92077\pi^6}{170100} + \frac{2503\zeta^2(3)}{45} \right) + \mathcal{O}(\epsilon^7)\,,
\end{aligned}
$$

$$
\begin{aligned}
F_{71} &= \epsilon^4\left(H_{0,1,0,1}(x) - H_{0,0,1,1}(x) + \frac{\pi^2}{6}H_{0,1}(x) - \frac{\pi^4}{30}\right)\\
&+ \epsilon^5\Big(-2H_{0,0,0,0,1}(x) - 2H_{0,0,0,1,1}(x) - 2H_{0,0,1,0,1}(x) - 10H_{0,0,1,1,1}(x)\\
&+ 2H_{0,1,0,1,1}(x) + 6H_{0,1,1,0,1}(x) - \frac{\pi^2}{6}H_{0,0,1}(x) + \pi^2 H_{0,1,1}(x) + 2\zeta(3)H_{0,1}(x)\\
&- \frac{7\pi^2\zeta(3)}{6} - \zeta(5)\Big)\\
&+ \epsilon^6\Big(-\left(2\zeta(5) + \frac{\pi^2\zeta(3)}{3}\right)H_1(x) + \frac{9\pi^4}{40}H_{0,1}(x)\\
&+ \zeta(3)(-13H_{0,0,1}(x) + 9H_{0,1,1}(x) - 2H_{1,0,1}(x)) - \pi^2\Big(-H_{0,0,0,1}(x) - \frac{5}{6}H_{0,0,1,1}(x)\\
&+ H_{0,1,0,1}(x) + 6H_{0,1,1,1}(x) + \frac{1}{3}H_{1,0,0,1}(x)\Big) - 11H_{0,0,0,0,0,1}(x) - 11H_{0,0,0,0,1,1}(x)\\
&- 20H_{0,0,0,1,0,1}(x) - 20H_{0,0,0,1,1,1}(x) - 16H_{0,0,1,0,0,1}(x) - 26H_{0,0,1,0,1,1,}(x)\\
&- 29H_{0,0,1,1,0,1}(x) - 76H_{0,0,1,1,1,1}(x) - 14H_{0,1,0,0,0,1}(x) - 12H_{0,1,0,0,1,1}(x)\\
&+ 2H_{0,1,0,1,0,1}(x) - 4H_{0,1,0,1,1,1}(x) + 3H_{0,1,1,0,0,1}(x) + 12H_{0,1,1,0,1,1}(x)\\
&+ 36H_{0,1,1,1,0,1}(x) + 4H_{1,0,0,0,0,1}(x) + 2H_{1,0,0,1,0,1}(x) + 2H_{1,0,1,0,0,1}(x)\\
&- \frac{1219\pi^6}{15120}\Big) + \mathcal{O}(\epsilon^7), \qquad (14)
\end{aligned}
$$

H are Harmonic Polylogarithms

$$
\begin{aligned}
\mathrm{H}(0;x) &= \ln x\,,\\
\mathrm{H}(1;x) &= \int_0^x \frac{dx'}{1-x'} = -\ln(1-x)\\
\mathrm{H}(-1;x) &= \int_0^x \frac{dx'}{1+x'} = \ln(1+x)\,.
\end{aligned}
$$

$$
\mathrm{H}(\vec{m}_w;x) = \int_0^x dx'\; f(a;x')\; \mathrm{H}(\vec{m}_{w-1};x')\,.
$$

Check: (s=-1.3,m=1.0)

$$I^{\text{analytic}}_{1,1,0,1,1,0,1,1,1,1,-1,1} = \frac{0.00078765}{\epsilon^6} - \frac{0.00393624}{\epsilon^5} + \frac{0.0190587}{\epsilon^4} - \frac{0.0151068}{\epsilon^3} + \frac{0.290244}{\epsilon^2} + \frac{1.37654}{\epsilon} + 4.82542,$$

$$I^{\text{analytic}}_{1,1,0,1,1,-1,1,1,1,1,-1,1} = \frac{-6.69426}{\epsilon} - 63.1207.$$

$$I^{\text{numeric}}_{1,1,0,1,1,0,1,1,1,1,-1,1} = \frac{0.000788}{\epsilon^6} - \frac{0.003936}{\epsilon^5} + \frac{0.019058 \pm 0.000002}{\epsilon^4} - \frac{0.015109 \pm 0.000035}{\epsilon^3} + \frac{0.290192 \pm 0.000756}{\epsilon^2} + \frac{1.37606 \pm 0.01581}{\epsilon} + 4.80886 \pm 0.31758,$$

$$I^{\text{numeric}}_{1,1,0,1,1,-1,1,1,1,1,-1,1} = \frac{-6.69429 \pm 0.00003}{\epsilon} - 63.1213 \pm 0.0004,$$

FIESTA packages

Thanks!