# 中微子振荡的物质效应与重正化群方程

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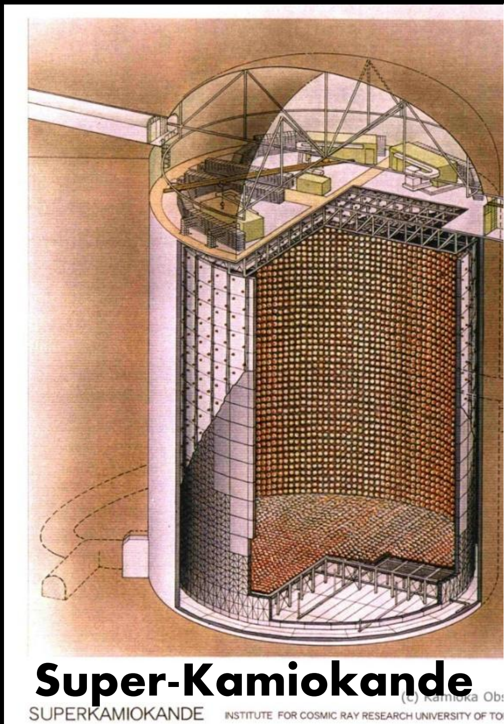
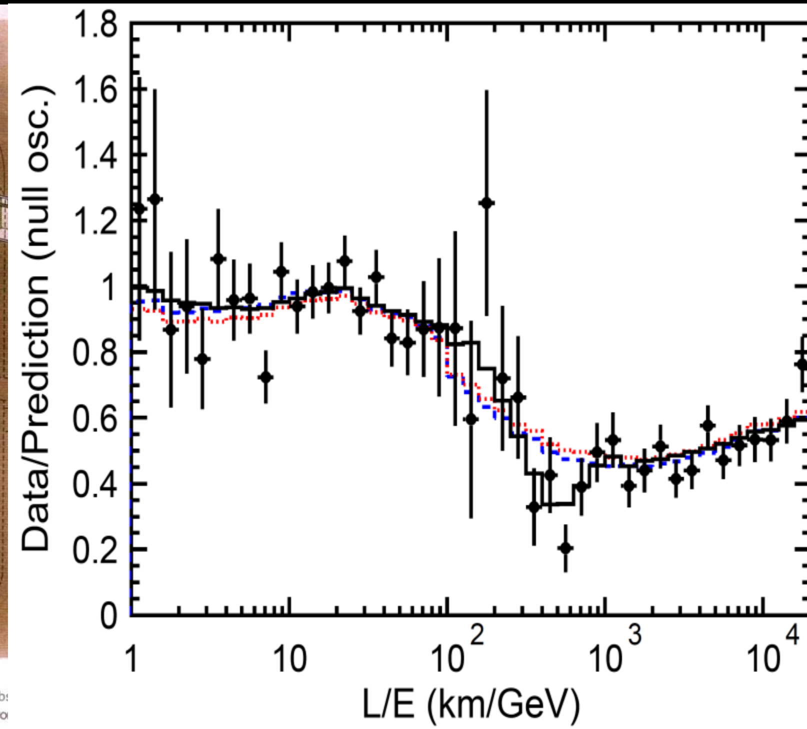
# Discovery of Neutrino Oscillations

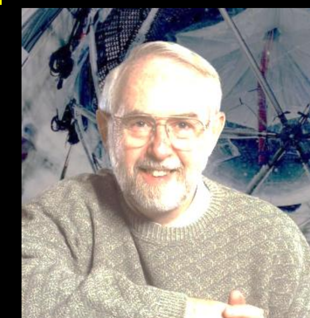
01



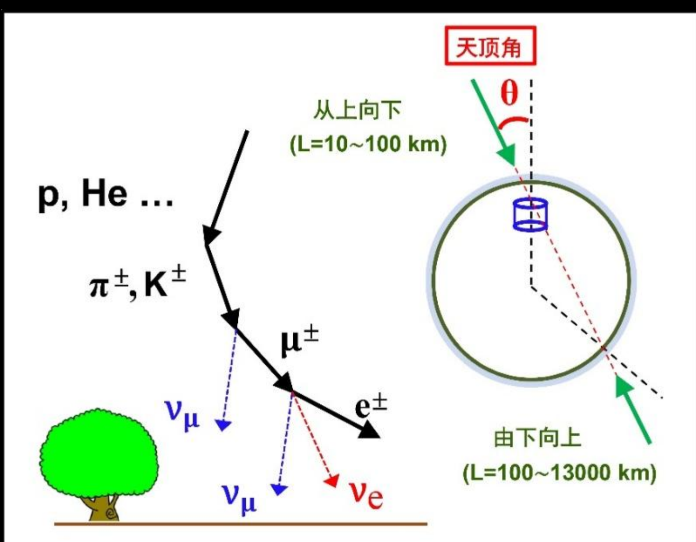



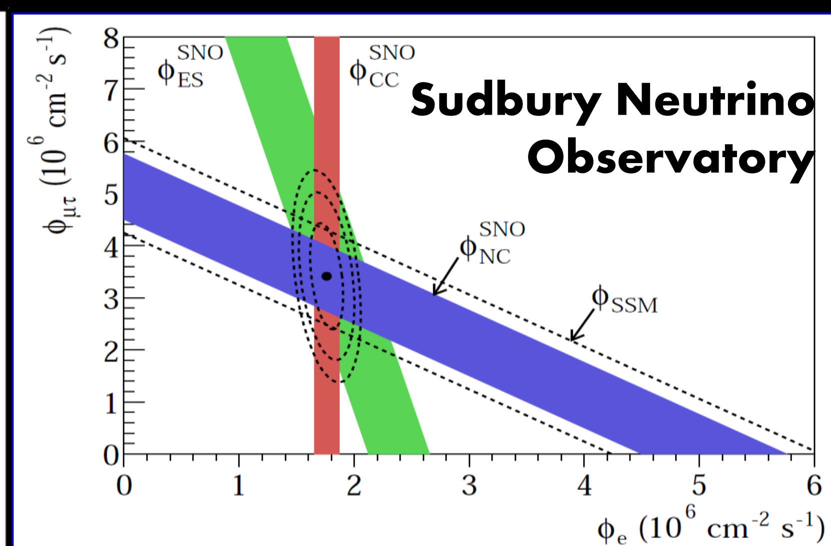

Takaaki Kajita

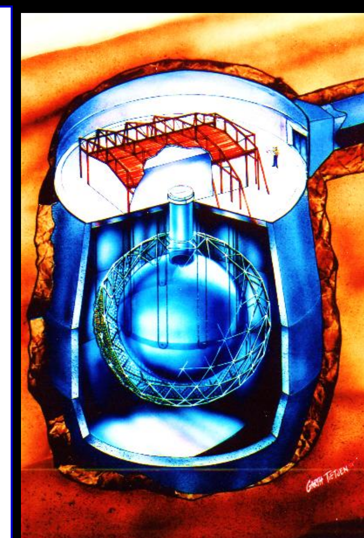
**Nobel Prize in Physics 15**



Arthur McDonald



# Status of Neutrino Oscillations



$m_1 < m_2 < m_3$ (NO) or $m_3 < m_1 < m_2$ (IO)

NuFIT 4.0 (2018)

without SK atmospheric data

| | Normal Ordering (best fit) | | Inverted Ordering ($\Delta\chi^2 = 4.7$) | |
|---|---|---|---|---|
| | bfp $\pm 1\sigma$ | $3\sigma$ range | bfp $\pm 1\sigma$ | $3\sigma$ range |
| $\sin^2\theta_{12}$ | $0.310^{+0.013}_{-0.012}$ | $0.275 \to 0.350$ | $0.310^{+0.013}_{-0.012}$ | $0.275 \to 0.350$ |
| $\theta_{12}/^\circ$ | $33.82^{+0.78}_{-0.76}$ | $31.61 \to 36.27$ | $33.82^{+0.78}_{-0.76}$ | $31.61 \to 36.27$ |
| $\sin^2\theta_{23}$ | $0.580^{+0.017}_{-0.021}$ | $0.418 \to 0.627$ | $0.584^{+0.016}_{-0.020}$ | $0.423 \to 0.629$ |
| $\theta_{23}/^\circ$ | $49.6^{+1.0}_{-1.2}$ | $40.3 \to 52.4$ | $49.8^{+1.0}_{-1.1}$ | $40.6 \to 52.5$ |
| $\sin^2\theta_{13}$ | $0.02241^{+0.00065}_{-0.00065}$ | $0.02045 \to 0.02439$ | $0.02264^{+0.00066}_{-0.00066}$ | $0.02068 \to 0.02463$ |
| $\theta_{13}/^\circ$ | $8.61^{+0.13}_{-0.13}$ | $8.22 \to 8.99$ | $8.65^{+0.13}_{-0.13}$ | $8.27 \to 9.03$ |
| $\delta_{CP}/^\circ$ | $215^{+40}_{-29}$ | $125 \to 392$ | $284^{+27}_{-29}$ | $196 \to 360$ |
| $\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$ | $7.39^{+0.21}_{-0.20}$ | $6.79 \to 8.01$ | $7.39^{+0.21}_{-0.20}$ | $6.79 \to 8.01$ |
| $\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$ | $+2.525^{+0.033}_{-0.032}$ | $+2.427 \to +2.625$ | $-2.512^{+0.034}_{-0.032}$ | $-2.611 \to -2.412$ |

**Neutrino mass ordering: normal ordering favored at the 2~3σ C.L.**

# Status of Neutrino Oscillations



$m_1 < m_2 < m_3$ (NO) or $m_3 < m_1 < m_2$ (IO)

NuFIT 4.0 (2018)

with SK atmospheric data

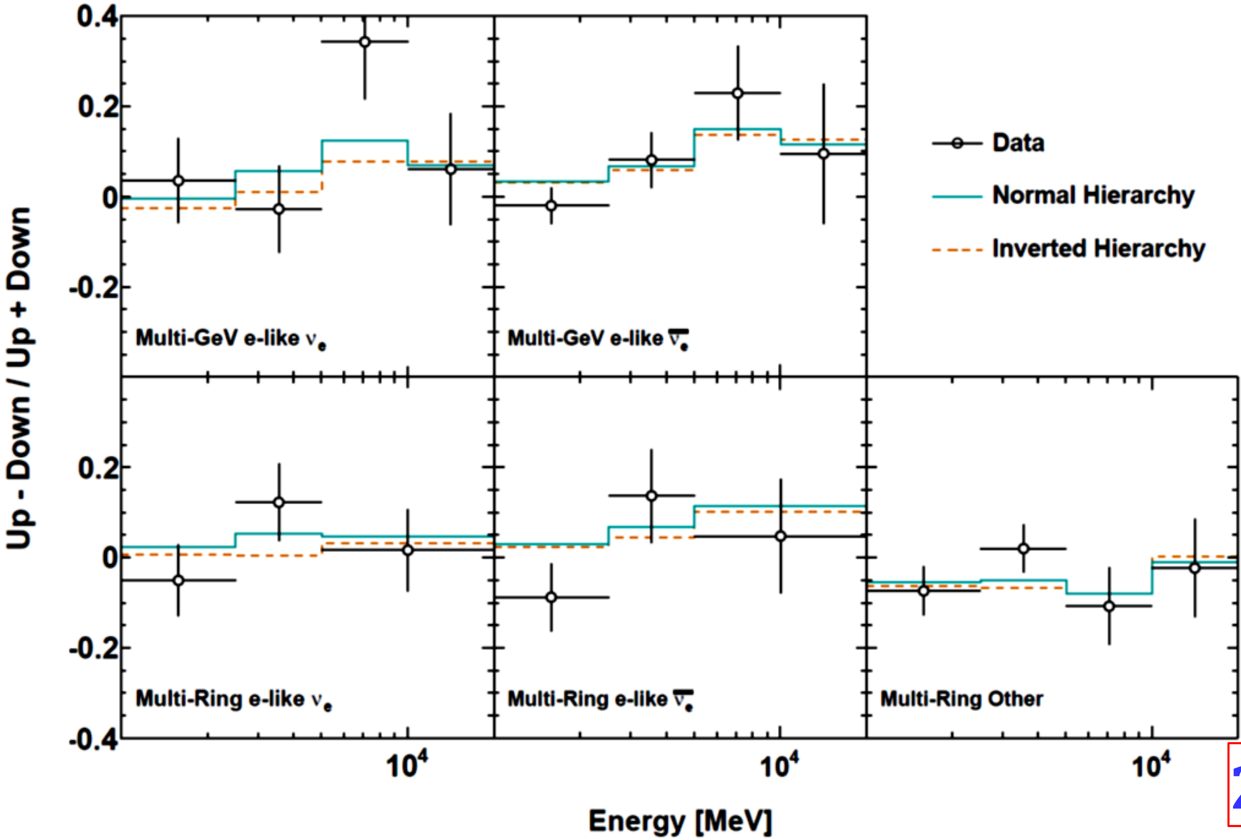
| | Normal Ordering (best fit) | | Inverted Ordering ($\Delta\chi^2 = 9.3$) | |
|---|---|---|---|---|
| | bfp $\pm 1\sigma$ | $3\sigma$ range | bfp $\pm 1\sigma$ | $3\sigma$ range |
| $\sin^2\theta_{12}$ | $0.310^{+0.013}_{-0.012}$ | $0.275 \to 0.350$ | $0.310^{+0.013}_{-0.012}$ | $0.275 \to 0.350$ |
| $\theta_{12}/^\circ$ | $33.82^{+0.78}_{-0.76}$ | $31.61 \to 36.27$ | $33.82^{+0.78}_{-0.75}$ | $31.62 \to 36.27$ |
| $\sin^2\theta_{23}$ | $0.582^{+0.015}_{-0.019}$ | $0.428 \to 0.624$ | $0.582^{+0.015}_{-0.018}$ | $0.433 \to 0.623$ |
| $\theta_{23}/^\circ$ | $49.7^{+0.9}_{-1.1}$ | $40.9 \to 52.2$ | $49.7^{+0.9}_{-1.0}$ | $41.2 \to 52.1$ |
| $\sin^2\theta_{13}$ | $0.02240^{+0.00065}_{-0.00066}$ | $0.02044 \to 0.02437$ | $0.02263^{+0.00065}_{-0.00066}$ | $0.02067 \to 0.02461$ |
| $\theta_{13}/^\circ$ | $8.61^{+0.12}_{-0.13}$ | $8.22 \to 8.98$ | $8.65^{+0.12}_{-0.13}$ | $8.27 \to 9.03$ |
| $\delta_{CP}/^\circ$ | $217^{+40}_{-28}$ | $135 \to 366$ | $280^{+25}_{-28}$ | $196 \to 351$ |
| $\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$ | $7.39^{+0.21}_{-0.20}$ | $6.79 \to 8.01$ | $7.39^{+0.21}_{-0.20}$ | $6.79 \to 8.01$ |
| $\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$ | $+2.525^{+0.033}_{-0.031}$ | $+2.431 \to +2.622$ | $-2.512^{+0.034}_{-0.031}$ | $-2.606 \to -2.413$ |

Neutrino mass ordering: normal ordering favored at the 2~3σ C.L.

# Sensitivity to Mass Ordering



SK atmospheric preference for NO due to excess of $e$-like events




**C. Giunti, NNN 2018**

Super-Kamiokande

$$\chi^2_{\text{IO}} - \chi^2_{\text{NO}} = 4.33$$

[PRD 97 (2018) 072001]

**MSW Resonance**

$$\boxed{2V_e E = \Delta m^2_{31} \cos 2\theta_{13}}$$

**Global-fit analysis**

**NO:** $\Delta m^2_{31} > 0$

**IO:** $\Delta m^2_{31} < 0$

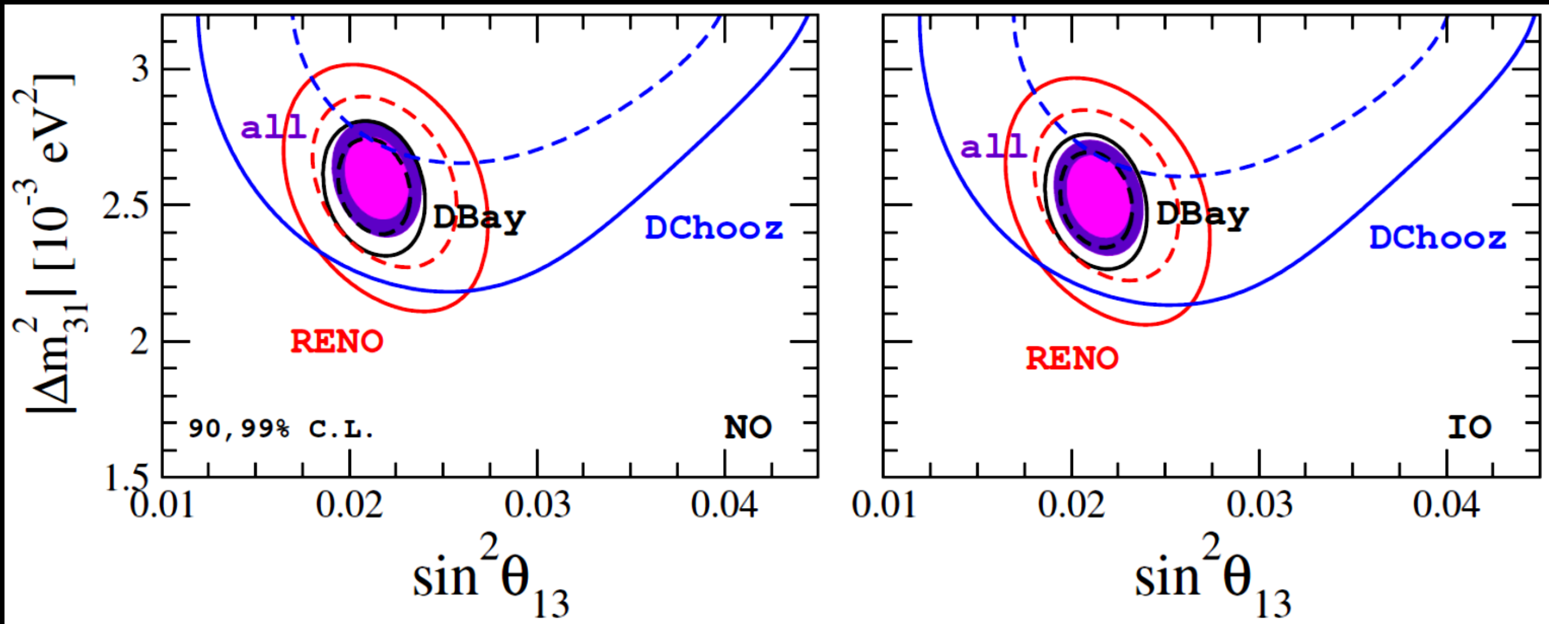
$\nu_\mu \rightleftharpoons \nu_e$

**Resonant enhancement**

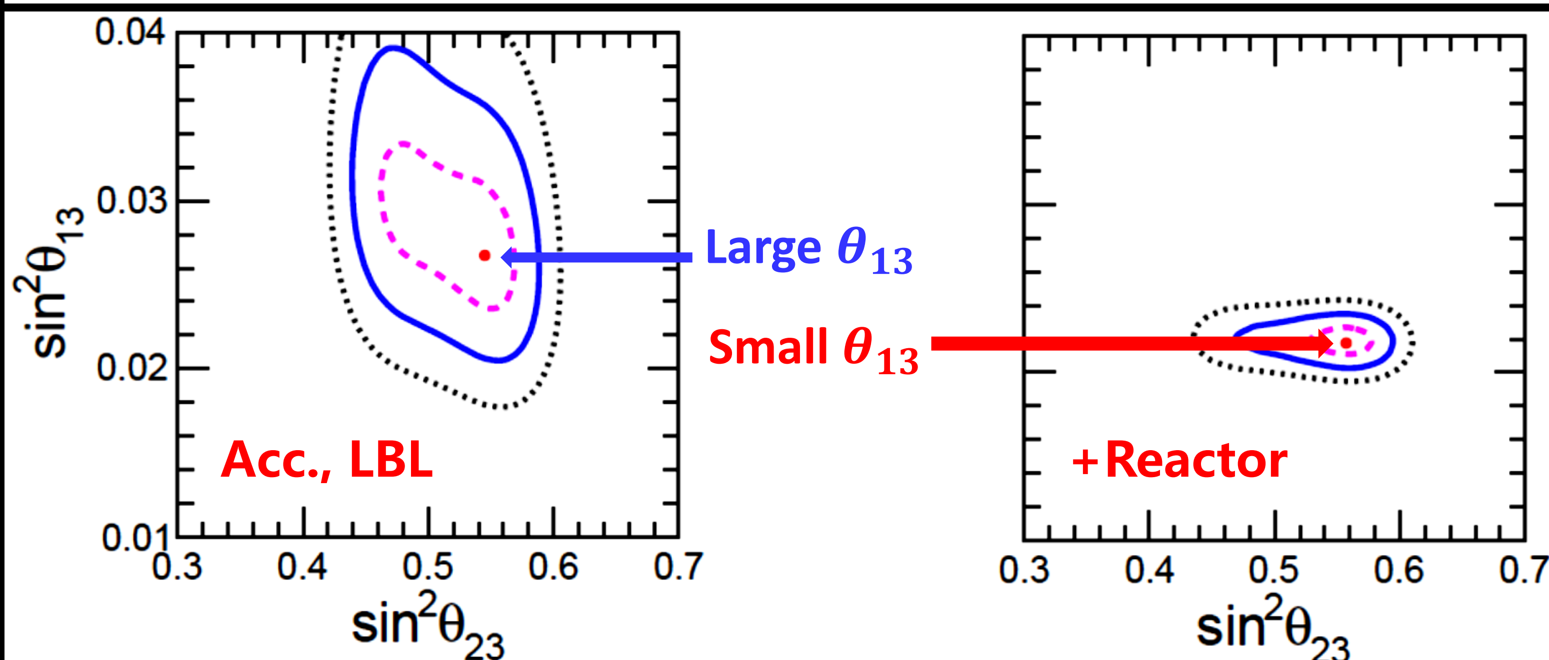
| | | |
|---|---|---|
| Bari: | $\chi^2_{\text{IO}} - \chi^2_{\text{NO}} = 9.5$ | $(\approx 3.1\sigma)$ |
| NuFit: | $\chi^2_{\text{IO}} - \chi^2_{\text{NO}} = 9.1$ | $(\approx 3.0\sigma)$ |
| Valencia: | $\chi^2_{\text{IO}} - \chi^2_{\text{NO}} = 11.7$ | $(\approx 3.4\sigma)$ |

# Sensitivity to Mass Ordering






Sensitivity to $\theta_{13}$ is dominated by DYB

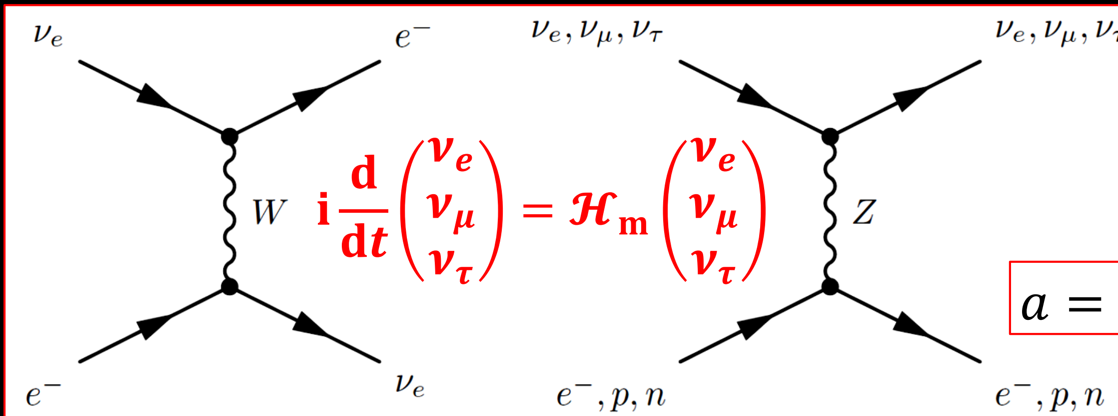



Large $\theta_{13}$ is preferred by T2K/NOvA

Small $\theta_{13}$ is preferred by DYB/RENO

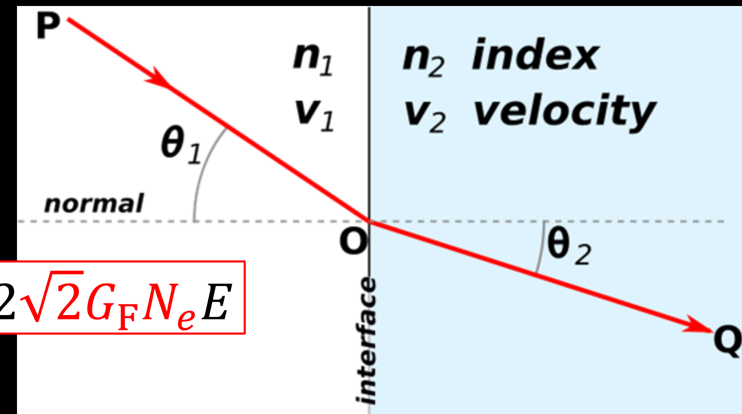
# Matter Effects on Neutrino Oscillations






$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \mathcal{H}_\mathrm{m}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$$

$$a = 2\sqrt{2}G_\mathrm{F}N_e E$$




$$\mathcal{H}_\mathrm{m} = \frac{1}{2E}\left[U\begin{pmatrix}m_1^2 & 0 & 0\\ 0 & m_2^2 & 0\\ 0 & 0 & m_3^2\end{pmatrix}U^\dagger + \begin{pmatrix}a & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right] \Rightarrow V^\dagger\mathcal{H}_\mathrm{m}V = \frac{1}{2E}\begin{pmatrix}\widetilde{m}_1^2 & 0 & 0\\ 0 & \widetilde{m}_2^2 & 0\\ 0 & 0 & \widetilde{m}_3^2\end{pmatrix}$$

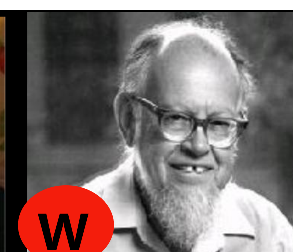
$$P(\nu_\mu \to \nu_e) \simeq \sin^2 2\theta_{13}\sin^2\theta_{23}\frac{\sin^2[(1-\widehat{A})\Delta]}{(1-\widehat{A})^2} \qquad \Delta \equiv \frac{\Delta m_{31}^2 L}{4E} \qquad \widehat{A} \equiv \frac{a}{\Delta m_{31}^2}$$

Large $\theta_{13}$ from T2K/NOvA

$$+\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\sin 2\theta_{13}\sin 2\theta_{12}\sin 2\theta_{23}\cos(\Delta+\delta)\frac{\sin(\widehat{A}\Delta)}{\widehat{A}}\frac{\sin[(1-\widehat{A})\Delta]}{1-\widehat{A}}$$

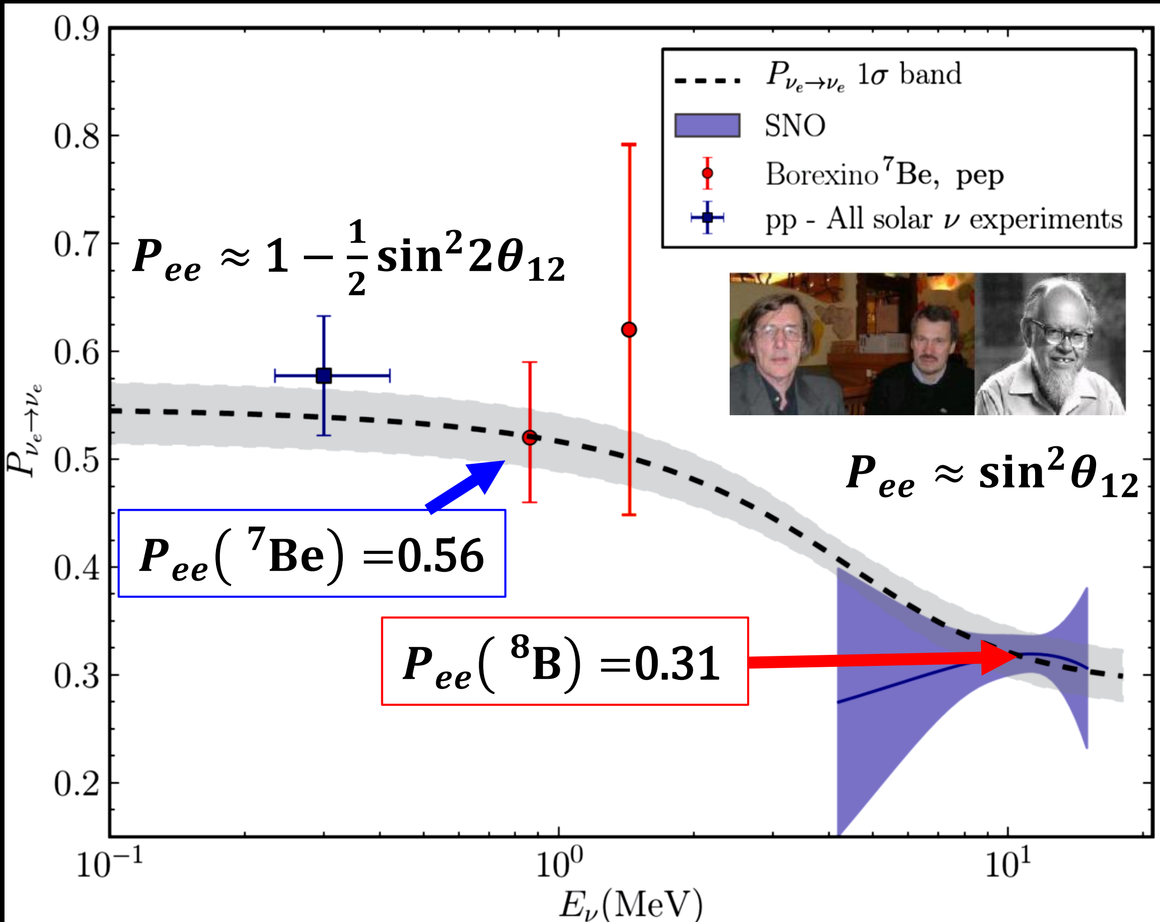
## Sensitivity to Mass Ordering

When other parameters are fixed, the NO will be favored to realize a smaller value of $\theta_{13}$

# Matter Effects on Neutrino Oscillations






**For high-energy $^8$B neutrinos**

**at production $r = 0$**

$$\begin{pmatrix} |\tilde{\nu}_1(0)\rangle \\ |\tilde{\nu}_2(0)\rangle \end{pmatrix} = \begin{pmatrix} c_{\hat{\theta}} & -s_{\hat{\theta}} \\ s_{\hat{\theta}} & c_{\hat{\theta}} \end{pmatrix} \begin{pmatrix} |\nu_e(0)\rangle \\ |\nu_\mu(0)\rangle \end{pmatrix}$$

**adiabatic evolution**

$$\begin{pmatrix} |\tilde{\nu}_1(R)\rangle \\ |\tilde{\nu}_2(R)\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_1(0)\rangle \\ |\tilde{\nu}_2(0)\rangle \end{pmatrix}$$

**on the solar surface $r = R$**

$$\begin{pmatrix} |\nu_e(R)\rangle \\ |\nu_\mu(R)\rangle \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_1(R)\rangle \\ |\tilde{\nu}_2(R)\rangle \end{pmatrix}$$

**survival probability**

$$P_{ee} = c_{\hat{\theta}}^2 c_\theta^2 + s_{\hat{\theta}}^2 s_\theta^2 \longrightarrow \sin^2\theta$$

$$\hat{\theta} \longrightarrow \pi/2 \qquad \text{as } A \gg \Delta m^2$$

**For the MSW resonance to happen**

$$\boxed{\theta \in [0, \frac{\pi}{2}] \quad \Delta m_{21}^2 > 0} \Longleftrightarrow \boxed{\theta \in [0, \frac{\pi}{4}] \quad \Delta m_{21}^2}$$

$$\theta_{12} = 34^\circ$$

**Normal neutrino mass ordering**

**For low-energy $^7$Be neutrinos**

$$P_{ee} \approx 1 - \frac{1}{2}\sin^2 2\theta$$

**Oscillations in vacuum**

# Matter Effects on Neutrino Oscillations



## An example for two-flavor neutrino mixing

$$\mathcal{H}_{\rm m} = \frac{1}{2E} U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger + \begin{pmatrix} V_e & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2E} \left[ U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right]$$

The effective Hamiltonian in a more compact form:

$$\mathcal{H}_{\rm m} = \frac{1}{4E} \left[ U \begin{pmatrix} -\Delta m_{21}^2 & 0 \\ 0 & +\Delta m_{21}^2 \end{pmatrix} U^\dagger + \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right] + \frac{m_1^2 + m_2^2 + a}{4E}$$

$$\Delta m_{21}^2 \equiv m_2^2 - m_1^2$$

$$\Delta \widetilde{m}_{21}^2 \equiv \widetilde{m}_2^2 - \widetilde{m}_1^2$$

$$\mathcal{H}_{\rm m} = \frac{1}{4E} \begin{pmatrix} a - \Delta m_{21}^2 c_{2\theta} & \Delta m_{21}^2 s_{2\theta} \\ \Delta m_{21}^2 s_{2\theta} & \Delta m_{21}^2 c_{2\theta} - a \end{pmatrix}$$

Converted into the mass basis

$$\mathcal{H}_{\rm m} = \frac{1}{4E} \left[ V \begin{pmatrix} -\Delta \widetilde{m}_{21}^2 & 0 \\ 0 & +\Delta \widetilde{m}_{21}^2 \end{pmatrix} V^\dagger \right]$$

Mixing matrix & mass states in matter

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} c_{\widetilde{\theta}} & s_{\widetilde{\theta}} \\ -s_{\widetilde{\theta}} & c_{\widetilde{\theta}} \end{pmatrix} \begin{pmatrix} |\widetilde{\nu}_1\rangle \\ |\widetilde{\nu}_2\rangle \end{pmatrix} \equiv \widetilde{U} \begin{pmatrix} |\widetilde{\nu}_1\rangle \\ |\widetilde{\nu}_2\rangle \end{pmatrix}$$

$$\Delta \widetilde{m}_{21}^2 = \sqrt{\left(\Delta m_{21}^2 c_{2\theta} - a\right)^2 + \left(\Delta m_{21}^2 s_{2\theta}\right)^2} \qquad s_{2\widetilde{\theta}}^2 = \frac{\Delta m_{21}^2 s_{2\theta}^2}{\left(\Delta m_{21}^2 c_{2\theta} - a\right)^2 + \left(\Delta m_{21}^2 s_{2\theta}\right)^2}$$

Relationship between the mixing angle (mass difference) in vacuum and that in matter

# Matter Effects on Neutrino Oscillations



## A: Establish the relationship between intrinsic and effective parameters

$$\mathcal{H}_{\rm m} = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \equiv \frac{\Omega_{\rm m}}{2E}$$

$$\Omega_{\rm v} = U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}U^\dagger$$

$$D_l = \begin{pmatrix} m_e^2 & 0 & 0 \\ 0 & m_\mu^2 & 0 \\ 0 & 0 & m_\tau^2 \end{pmatrix}$$

$$\Omega_{\rm m} = V\begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix}V^\dagger$$

**Calculate the commutators:**

$$[D_l, \Omega_{\rm m}] \equiv i\widetilde{X}_l$$

$$[D_l, \Omega_{\rm v}] \equiv iX_l$$

$$X_l = i\begin{pmatrix} 0 & \Delta_{e\mu}Z_{\mu e} & \Delta_{\tau e}Z_{e\tau} \\ \Delta_{\mu e}Z_{e\mu} & 0 & \Delta_{\mu\tau}Z_{\tau\mu} \\ \Delta_{e\tau}Z_{\tau e} & \Delta_{\tau\mu}Z_{\mu\tau} & 0 \end{pmatrix}$$

$$\widetilde{X}_l = i\begin{pmatrix} 0 & \Delta_{e\mu}\widetilde{Z}_{\mu e} & \Delta_{\tau e}\widetilde{Z}_{e\tau} \\ \Delta_{\mu e}\widetilde{Z}_{e\mu} & 0 & \Delta_{\mu\tau}\widetilde{Z}_{\tau\mu} \\ \Delta_{e\tau}\widetilde{Z}_{\tau e} & \Delta_{\tau\mu}\widetilde{Z}_{\mu\tau} & 0 \end{pmatrix}$$

$$\Delta_{\alpha\beta} \equiv m_\alpha^2 - m_\beta^2$$

$$\Delta_{ij} \equiv m_i^2 - m_j^2$$

$$\tilde{Z}_{\alpha\beta} \equiv \sum_i \widetilde{m}_i^2\, V_{\alpha i}V_{\beta i}^*$$

$$Z_{\alpha\beta} \equiv \sum_i m_i^2\, U_{\alpha i}U_{\beta i}^*$$

Harrison & Scott, PLB, 00;
Xing, PLB, 00

$$\mathrm{Det}[X_l] = 2i\Delta_{\mu e}\Delta_{\tau e}\Delta_{\tau\mu}\mathrm{Im}\left[Z_{\mu e}Z_{\tau e}Z_{\mu\tau}\right] = 2i\Delta_{\mu e}\Delta_{\tau e}\Delta_{\tau\mu}\Delta_{21}\Delta_{31}\Delta_{32}\mathcal{J}$$

$$\mathrm{Det}\left[\widetilde{X}_l\right] = 2i\Delta_{\mu e}\Delta_{\tau e}\Delta_{\tau\mu}\mathrm{Im}\left[\tilde{Z}_{\mu e}\tilde{Z}_{\tau e}\tilde{Z}_{\mu\tau}\right] = 2i\Delta_{\mu e}\Delta_{\tau e}\Delta_{\tau\mu}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}\tilde{\mathcal{J}}$$

$$\boxed{\frac{\tilde{\mathcal{J}}}{\mathcal{J}} = \frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}}$$

# Matter Effects on Neutrino Oscillations



## B: Series expansion of the effective parameters

**Eigenvalues:**

Freund, PRD, 01; Akhmedov et al., JHEP, 04

$$\widetilde{m}_1^2 = m_1^2 + \Delta_{31}\left(\hat{A} + \alpha s_{12}^2 + s_{13}^2\frac{\hat{A}}{\hat{A}-1} + \alpha^2\frac{\sin^2 2\theta_{12}}{4\hat{A}}\right)$$

$$\widetilde{m}_2^2 = m_1^2 + \Delta_{31}\left(\alpha c_{12}^2 - \alpha^2\frac{\sin^2 2\theta_{12}}{4\hat{A}}\right)$$

$$\alpha \equiv \frac{\Delta_{21}}{\Delta_{31}} \approx 0.03$$

$$s_{13}^2 \approx 0.02$$

up to the second order of $\alpha$ and $s_{13}^2$

$$\widetilde{m}_3^2 = m_1^2 + \Delta_{31}\left(1 - s_{13}^2\frac{\hat{A}}{\hat{A}-1}\right)$$

Divergent in the limits of $\hat{A} \to 0$ and $\hat{A} \to 1$

**Oscillation Probabilities:**

$$\widetilde{P}_{ee} = 1 - \alpha^2\sin^2 2\theta_{12}\frac{\sin^2\hat{A}\Delta}{\hat{A}^2} - 4s_{13}^2\frac{\sin^2(\hat{A}-1)\Delta}{(\hat{A}-1)^2}$$

Finite in the limits $\hat{A} \to 0$ and $\hat{A} \to 1$

$$\widetilde{P}_{e\mu} = \alpha^2\sin^2 2\theta_{12}c_{23}^2\frac{\sin^2\hat{A}\Delta}{\hat{A}^2} + 4s_{13}^2 s_{23}^2\frac{\sin^2(\hat{A}-1)\Delta}{(\hat{A}-1)^2}$$

$$+ 2\alpha\, s_{13}\sin 2\theta_{12}\,\sin 2\theta_{23}\,\cos(\Delta-\delta)\frac{\sin\hat{A}\Delta}{\hat{A}}\frac{\sin(\hat{A}-1)\Delta}{(\hat{A}-1)}$$

Valid only when $\Delta_{21}$-driven osci. are small $\alpha\Delta \ll 1$

$$\Delta \equiv \frac{\Delta_{31}L}{4E}$$

# A Differential Way to Understand Matter Effects



$$\mathcal{H}_{\rm m} = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2\end{pmatrix}U^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}\right] \Rightarrow \mathcal{H}_{\rm m} = \frac{1}{2E}\left[V\begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2\end{pmatrix}V^\dagger\right]$$

Take the derivative of the effective Hamiltonian with respect to ***a***

$$\dot{D} + \left[V^\dagger \dot{V}, D\right] = V^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix} V = \begin{pmatrix} |V_{e1}|^2 & V_{e1}^* V_{e2} & V_{e1}^* V_{e3} \\ V_{e2}^* V_{e1} & |V_{e2}|^2 & V_{e2}^* V_{e3} \\ V_{e3}^* V_{e1} & V_{e3}^* V_{e2} & |V_{e3}|^2\end{pmatrix}$$

$$D \equiv \begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2\end{pmatrix}$$

A complete set of differential equations ( “RGEs” )

$$\widetilde{\Delta}_{ij} \equiv \widetilde{m}_i^2 - \widetilde{m}_j^2$$

$$\frac{\rm d}{{\rm d}a}|V_{e1}|^2 = 2|V_{e1}|^2\left(|V_{e2}|^2\widetilde{\Delta}_{12}^{-1} - |V_{e3}|^2\widetilde{\Delta}_{31}^{-1}\right)$$
$$\frac{\rm d}{{\rm d}a}|V_{e2}|^2 = 2|V_{e2}|^2\left(|V_{e3}|^2\widetilde{\Delta}_{23}^{-1} - |V_{e1}|^2\widetilde{\Delta}_{12}^{-1}\right)$$
$$\frac{\rm d}{{\rm d}a}|V_{e3}|^2 = 2|V_{e3}|^2\left(|V_{e1}|^2\widetilde{\Delta}_{31}^{-1} - |V_{e2}|^2\widetilde{\Delta}_{23}^{-1}\right)$$

**Electron flavor is SPECIAL**

$$\frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{12} = |V_{e1}|^2 - |V_{e2}|^2$$
$$\frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{23} = |V_{e2}|^2 - |V_{e3}|^2$$
$$\frac{\rm d}{{\rm d}a}\widetilde{\Delta}_{31} = |V_{e3}|^2 - |V_{e1}|^2$$

S.H. Chiu & T.K. Kuo, PRD 2018; Z.Z. Xing, S. Zhou & Y.L. Zhou, JHEP 2018

# A Differential Way to Understand Matter Effects



**Diff. Invariant**

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[\ln\left(\boxed{|V_{e1}|^2|V_{e2}|^2|V_{e3}|^2\widetilde{\Delta}^2_{12}\widetilde{\Delta}^2_{23}\widetilde{\Delta}^2_{31}}\right)\right] = \sum_{i=1}^{3}\frac{\mathrm{d}}{\mathrm{d}a}\left(\ln|V_{ei}|^2\right) + \sum_{j>k}\frac{\mathrm{d}}{\mathrm{d}a}\left(\ln\widetilde{\Delta}^2_{jk}\right) = 0$$

$$\widetilde{\mathcal{J}} = \mathrm{Im}\left[V_{e1}V_{\mu 2}V^*_{e2}V^*_{\mu 1}\right]$$

**Jarlskog rephrasing invariant**

$$\frac{\mathrm{d}}{\mathrm{d}a}\ln\left[\boxed{\widetilde{\mathcal{J}}\widetilde{\Delta}_{12}\widetilde{\Delta}_{23}\widetilde{\Delta}_{31}}\right] = 0$$

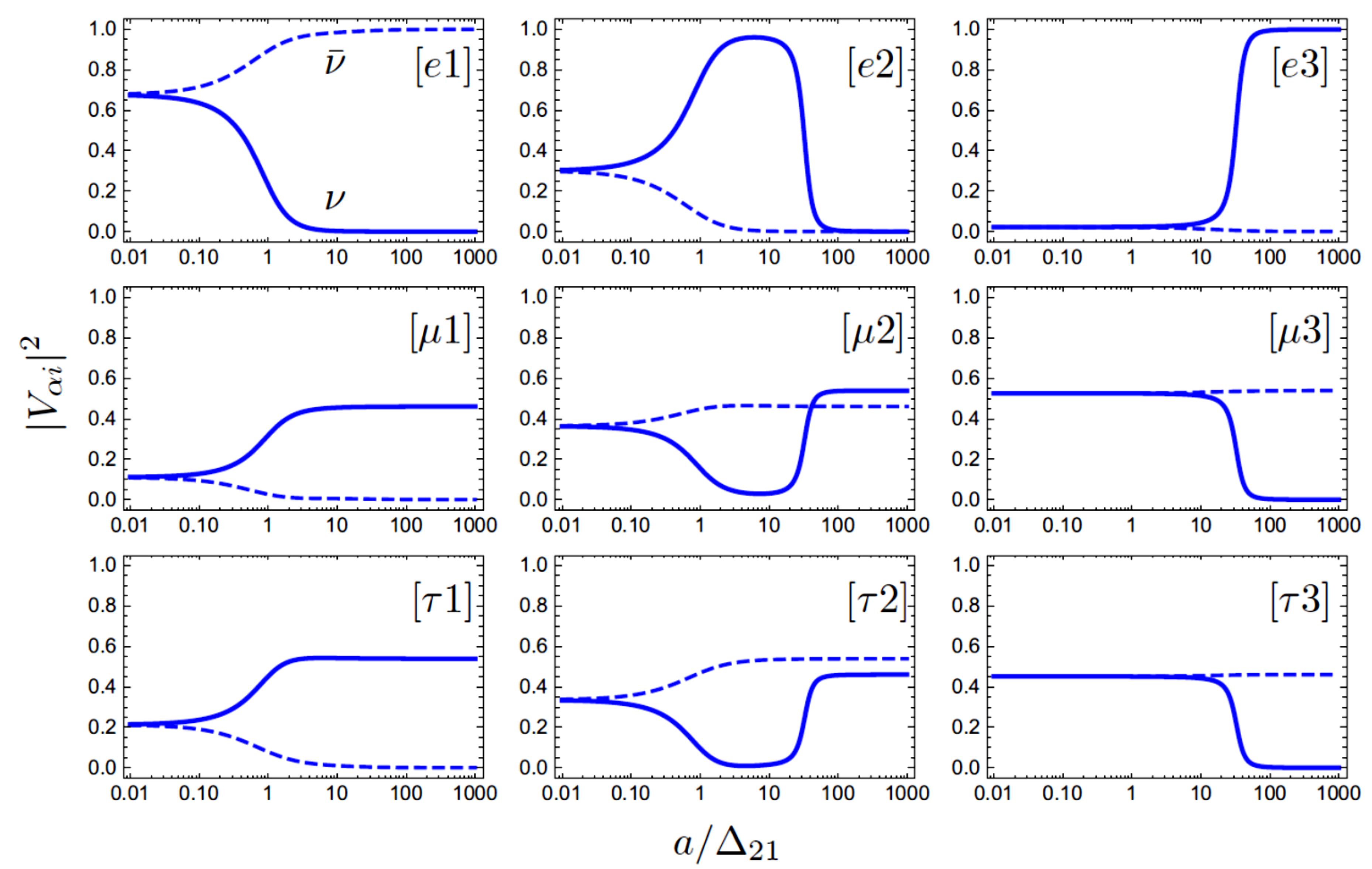
**Differential Invariant**

$$\begin{aligned}
\dot{V}_{e1} &= |V_{e2}|^2 V_{e1}\widetilde{\Delta}^{-1}_{12} - |V_{e3}|^2 V_{e1}\widetilde{\Delta}^{-1}_{31} + \sum_\alpha \dot{V}_{\alpha i}V^*_{\alpha i}V_{e1}\,, \\
\dot{V}^*_{e2} &= |V_{e3}|^2 V^*_{e2}\widetilde{\Delta}^{-1}_{23} - |V_{e1}|^2 V^*_{e2}\widetilde{\Delta}^{-1}_{12} + \sum_\alpha \dot{V}_{\alpha i}V^*_{\alpha i}V^*_{e2}\,, \\
\dot{V}^*_{\mu 1} &= V^*_{\mu 2}V^*_{e1}V_{e2}\widetilde{\Delta}^{-1}_{12} - V^*_{\mu 3}V^*_{e1}V_{e3}\widetilde{\Delta}^{-1}_{31} + \sum_\alpha \dot{V}_{\alpha i}V^*_{\alpha i}V^*_{\mu 1}\,, \\
\dot{V}_{\mu 2} &= V_{\mu 3}V_{e2}V^*_{e3}\widetilde{\Delta}^{-1}_{23} - V_{\mu 1}V_{e2}V^*_{e1}\widetilde{\Delta}^{-1}_{12} + \sum_\alpha \dot{V}_{\alpha i}V^*_{\alpha i}V_{\mu 2}\,.
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}|V_{\mu 1}|^2 &= |V_{\mu 1}|^2\left[\frac{|V_{e2}|^2}{\widetilde{\Delta}_{12}} - \frac{|V_{e3}|^2}{\widetilde{\Delta}_{31}}\right] + |V_{e1}|^2\left[\frac{|V_{\mu 2}|^2}{\widetilde{\Delta}_{12}} - \frac{|V_{\mu 3}|^2}{\widetilde{\Delta}_{31}}\right] - \left[\frac{|V_{\tau 3}|^2}{\widetilde{\Delta}_{12}} - \frac{|V_{\tau 2}|^2}{\widetilde{\Delta}_{31}}\right], \\
\frac{\mathrm{d}}{\mathrm{d}a}|V_{\mu 2}|^2 &= |V_{\mu 2}|^2\left[\frac{|V_{e3}|^2}{\widetilde{\Delta}_{23}} - \frac{|V_{e1}|^2}{\widetilde{\Delta}_{12}}\right] + |V_{e2}|^2\left[\frac{|V_{\mu 3}|^2}{\widetilde{\Delta}_{23}} - \frac{|V_{\mu 1}|^2}{\widetilde{\Delta}_{12}}\right] - \left[\frac{|V_{\tau 1}|^2}{\widetilde{\Delta}_{23}} - \frac{|V_{\tau 3}|^2}{\widetilde{\Delta}_{12}}\right], \\
\frac{\mathrm{d}}{\mathrm{d}a}|V_{\mu 3}|^2 &= |V_{\mu 3}|^2\left[\frac{|V_{e1}|^2}{\widetilde{\Delta}_{31}} - \frac{|V_{e2}|^2}{\widetilde{\Delta}_{23}}\right] + |V_{e3}|^2\left[\frac{|V_{\mu 1}|^2}{\widetilde{\Delta}_{31}} - \frac{|V_{\mu 2}|^2}{\widetilde{\Delta}_{23}}\right] - \left[\frac{|V_{\tau 2}|^2}{\widetilde{\Delta}_{31}} - \frac{|V_{\tau 1}|^2}{\widetilde{\Delta}_{23}}\right].
\end{aligned}$$

# Numerical Solutions to Mixing Matrix Elements






**Evolution of the mixing matrix elements in the NO case (input global-fit data)**

# Numerical Solutions to Mixing Parameters



**Standard Parametrization**

$$V = \begin{pmatrix} \tilde{c}_{13}\tilde{c}_{12} & \tilde{c}_{13}\tilde{s}_{12} & \tilde{s}_{13}e^{-i\tilde{\delta}} \\ -\tilde{s}_{12}\tilde{c}_{23} - \tilde{c}_{12}\tilde{s}_{23}\tilde{s}_{13}e^{i\tilde{\delta}} & +\tilde{c}_{12}\tilde{c}_{23} - \tilde{s}_{12}\tilde{s}_{23}\tilde{s}_{13}e^{i\tilde{\delta}} & \tilde{c}_{13}\tilde{s}_{23} \\ +\tilde{s}_{12}\tilde{s}_{23} - \tilde{c}_{12}\tilde{c}_{23}\tilde{s}_{13}e^{i\tilde{\delta}} & -\tilde{c}_{12}\tilde{s}_{23} - \tilde{s}_{12}\tilde{c}_{23}\tilde{s}_{13}e^{i\tilde{\delta}} & \tilde{c}_{13}\tilde{c}_{23} \end{pmatrix}$$

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\theta}_{12}}{\mathrm{d}a} &= \frac{1}{2}\sin 2\widetilde{\theta}_{12}\left(\cos^2\widetilde{\theta}_{13}\widetilde{\Delta}_{21}^{-1} - \sin^2\widetilde{\theta}_{13}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1}\right) , \\
\frac{\mathrm{d}\widetilde{\theta}_{13}}{\mathrm{d}a} &= \frac{1}{2}\sin 2\widetilde{\theta}_{13}\left(\cos^2\widetilde{\theta}_{12}\widetilde{\Delta}_{31}^{-1} + \sin^2\widetilde{\theta}_{12}\widetilde{\Delta}_{32}^{-1}\right) , \\
\frac{\mathrm{d}\widetilde{\theta}_{23}}{\mathrm{d}a} &= \frac{1}{2}\sin 2\widetilde{\theta}_{12}\sin\widetilde{\theta}_{13}\cos\widetilde{\delta}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1} , \\
\frac{\mathrm{d}\widetilde{\delta}}{\mathrm{d}a} &= -\sin 2\widetilde{\theta}_{12}\sin\widetilde{\theta}_{13}\sin\widetilde{\delta}\cot 2\widetilde{\theta}_{23}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1} ;
\end{aligned}$$

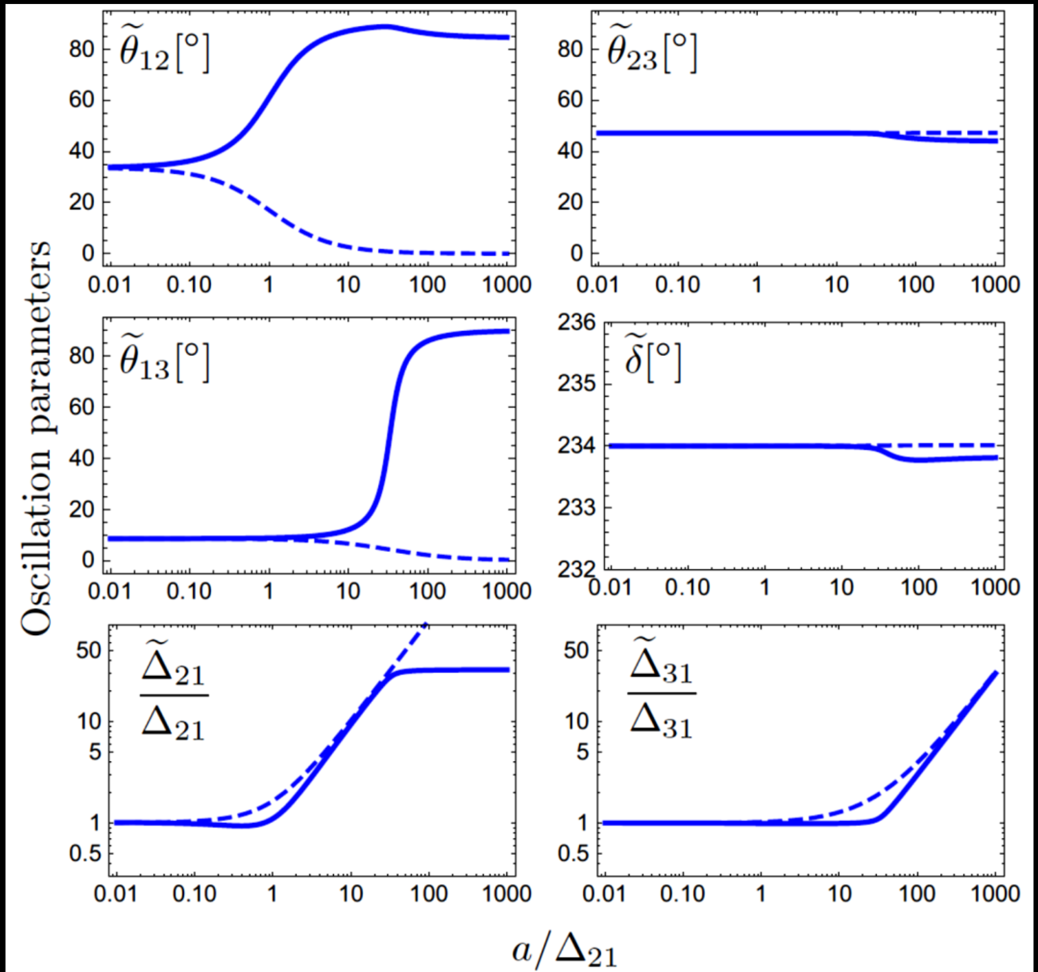
$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\Delta}_{21}}{\mathrm{d}a} &= -\cos^2\widetilde{\theta}_{13}\cos 2\widetilde{\theta}_{12} , \\
\frac{\mathrm{d}\widetilde{\Delta}_{31}}{\mathrm{d}a} &= \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{12} , \\
\frac{\mathrm{d}\widetilde{\Delta}_{32}}{\mathrm{d}a} &= \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\sin^2\widetilde{\theta}_{12} ,
\end{aligned}$$

$\{\widetilde{\theta}_{13}, \widetilde{\theta}_{12}\}$

$\{\widetilde{\Delta}_{21}, \widetilde{\Delta}_{31}, \widetilde{\Delta}_{32}\}$

⬇

$\{\widetilde{\theta}_{23}, \widetilde{\delta}\}$




**Evolution of $\{\widetilde{\theta}_{13}, \widetilde{\theta}_{12}\}$ and $\{\widetilde{\Delta}_{21}, \widetilde{\Delta}_{31}, \widetilde{\Delta}_{32}\}$ is independent of $\widetilde{\theta}_{23}$ and $\widetilde{\delta}$**

**Z.Z. Xing, S. Zhou & Y.L. Zhou, JHEP, 18**

**Are analytical solutions possible ???**

# Analytical Solutions to RGEs



## Series expansion of mass eigenvalues

$$\widetilde{\Delta}_{21} \approx \Delta_{31}\left[\frac{1}{2}\left(1+A-C_{13}\right)+\alpha\left(\frac{C_{13}+1-A\cos 2\theta_{13}}{2C_{13}}\sin^2\theta_{12}-\cos^2\theta_{12}\right)\right] ,$$

$$\widetilde{\Delta}_{31} \approx \Delta_{31}\left[\frac{1}{2}\left(1+A+C_{13}\right)+\alpha\left(\frac{C_{13}-1+A\cos 2\theta_{13}}{2C_{13}}\sin^2\theta_{12}-\cos^2\theta_{12}\right)\right] ,$$

$$\widetilde{\Delta}_{32} \approx \Delta_{31}\left[C_{13}+\alpha\sin^2\theta_{12}\left(\frac{A\cos 2\theta_{13}-1}{C_{13}}\right)\right] ,$$

Freund, PRD, 01;
Akhmedov et al., JHEP, 04

As a starting point

X. Wang, S. Zhou, to appear

Introduce a new mass-squared difference

$$\Delta_c \equiv \Delta_{31}\cos^2\theta_{12} + \Delta_{32}\sin^2\theta_{12} \qquad \alpha_c = \Delta_{21}/\Delta_c$$

Minakata, Parke, JHEP, 16;
Y.F. Li, J. Zhang, S. Zhou, J.Y. Zhu, JHEP, 16

$$\widetilde{\Delta}_{21} \approx \Delta_c\left[\frac{1}{2}\left(1+A_c-\widehat{C}_{13}\right)-\alpha_c\cos 2\theta_{12}\right] ,$$

$$\widetilde{\Delta}_{31} \approx \Delta_c\left[\frac{1}{2}\left(1+A_c+\widehat{C}_{13}\right)-\alpha_c\cos 2\theta_{12}\right] ,$$

$$\widetilde{\Delta}_{32} \approx \Delta_c\widehat{C}_{13} , \quad \boxed{\widehat{C}_{13} \equiv \sqrt{1-2A_c\cos 2\theta_{13}+A_c^2}}$$

$$\frac{d\widetilde{\Delta}_{31}}{da} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{12}$$

$$\frac{d\widetilde{\Delta}_{32}}{da} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\sin^2\widetilde{\theta}_{12}$$

$$\Rightarrow$$

$$\frac{1}{2}\left(1+\frac{A_c-\cos 2\theta_{13}}{\widehat{C}_{13}}\right) = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{12}$$

$$\frac{A_c-\cos 2\theta_{13}}{\widehat{C}_{13}} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\sin^2\widetilde{\theta}_{12}$$

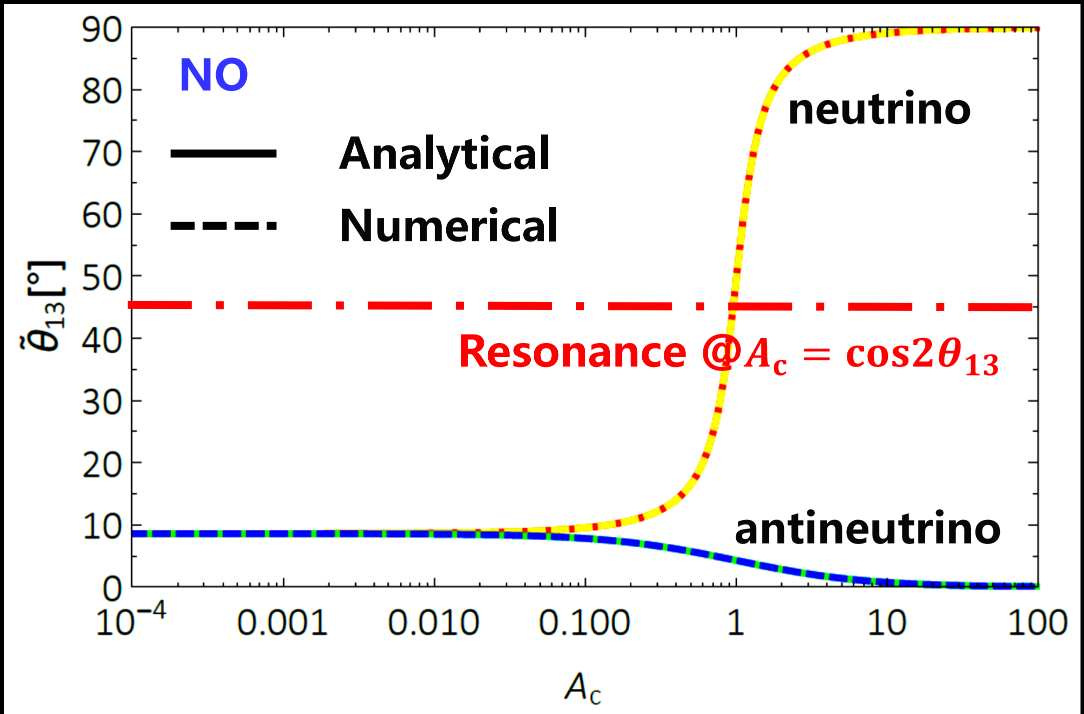
$$\cos^2\widetilde{\theta}_{13} = \frac{1}{2}\left(1-\frac{A_c-\cos 2\theta_{13}}{\widehat{C}_{13}}\right)$$

$$\sin^2 2\widetilde{\theta}_{13} = 1-\frac{(A_c-\cos 2\theta_{13})^2}{\widehat{C}_{13}^2} = \frac{\sin^2 2\theta_{13}}{(A_c-\cos 2\theta_{13})^2+\sin^2 2\theta_{13}}$$

**Note:** $\widetilde{\theta}_{13}$ is given by the formula in the limit of two-flavor neutrino mixing

# Analytical Solutions to RGEs






## Analytical result

$$\cos^2 \widetilde{\theta}_{13} = \frac{1}{2}\left(1 - \frac{A_c - \cos 2\theta_{13}}{\widehat{C}_{13}}\right)$$

$$\boxed{\widehat{C}_{13} \equiv \sqrt{1 - 2A_c \cos 2\theta_{13} + A_c^2}}$$

$$A_c = \frac{a}{\Delta_c}$$

**Order of magnitude**

$$= 6 \times 10^{-4} \left(\frac{N_e}{N_A\ \mathrm{cm}^{-3}}\right)\left(\frac{E}{10\ \mathrm{MeV}}\right)$$

**The Earth, $A_c$~1 for $E$ = 10 GeV**

## Analytical solution to $\widetilde{\theta}_{12}$

- Series expansion invalid for small values of $A_c$
- Introduce two functions of $A_c$ to be solved from RGEs

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}A_c} = -\frac{(\cos 2\theta_{12} + \mathcal{F})\cos^2\theta_{13}}{(\cos 2\theta_{12} + 2\mathcal{F})\alpha_c + A_c}$$

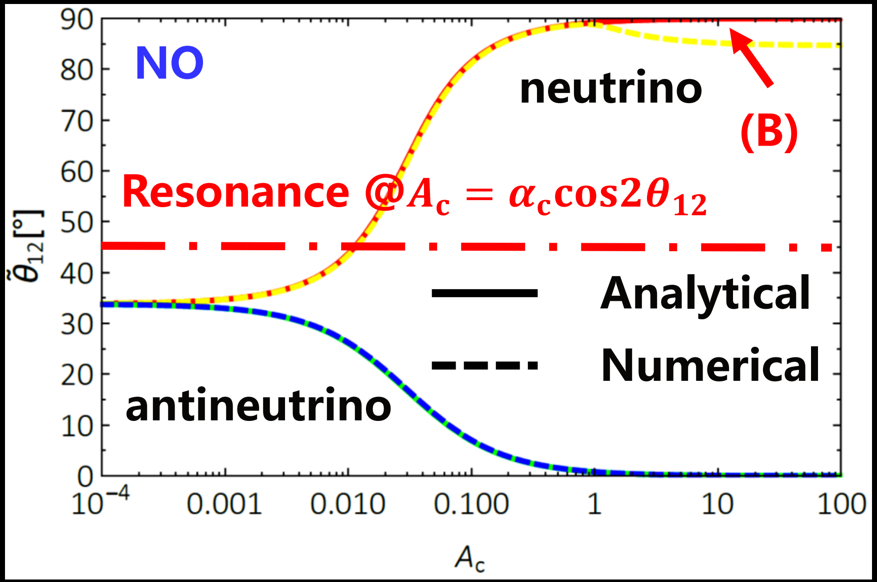
$$\widetilde{\Delta}_{21} = \Delta_c \left[\frac{1}{2}(1 + A_c - \widehat{C}_{13}) + \alpha_c(\underline{\mathcal{F} - \mathcal{G}})\right],$$

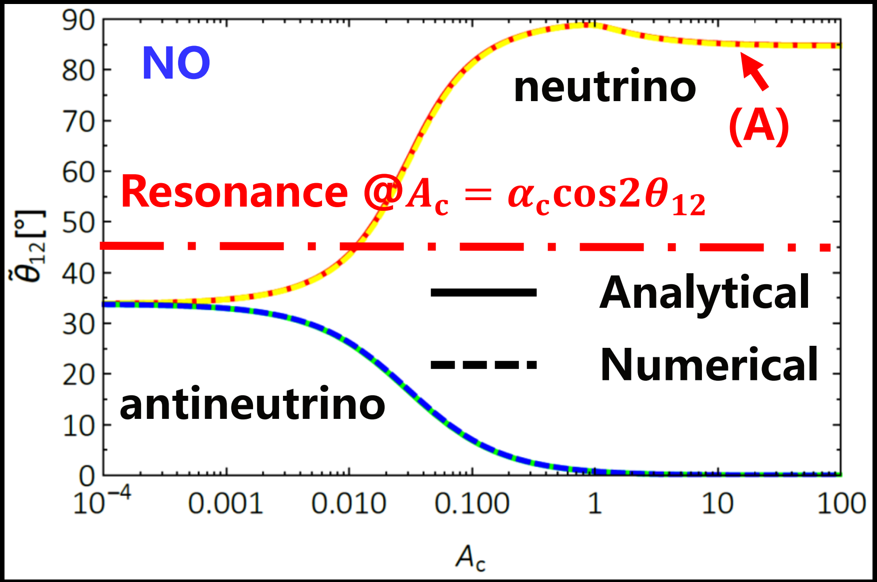
$$\widetilde{\Delta}_{31} = \Delta_c \left[\frac{1}{2}(1 + A_c + \widehat{C}_{13}) + \alpha_c\underline{\mathcal{F}}\right],$$

$$\widetilde{\Delta}_{32} = \Delta_c \left(\widehat{C}_{13} + \alpha_c\underline{\mathcal{G}}\right),$$

$$\boxed{\begin{aligned} &\mathrm{d}\mathcal{F}/\mathrm{d}A_c + \mathrm{d}\mathcal{G}/\mathrm{d}A_c = 0 \\ &\mathcal{F} + \mathcal{G} = -\cos 2\theta_{12} \end{aligned}}$$

# Analytical Solutions to RGEs




**Analytical Solution:**

$$\mathcal{F}(A_c) = \frac{1}{2\alpha_c}\left[\sqrt{(A_c - \alpha_c \cos 2\theta_{12})^2 + \alpha_c^2 \sin^2 2\theta_{12}} - (A_c + \alpha_c \cos 2\theta_{12})\right]$$

**(A)**
$$\cos^2 \tilde{\theta}_{12} = \frac{1}{2}\left(1 - \frac{A_* - \cos 2\theta_{12}}{\widehat{C}_{12}}\right) \boxed{\frac{2\widehat{C}_{13}\cos^2\theta_{13}}{\widehat{C}_{13} - A_c + \cos 2\theta_{13}}}$$

**Define** $A_* \equiv A_c/\alpha_c = a/\Delta_{21}$

$$\widehat{C}_{12} \equiv \sqrt{1 - 2A_* \cos 2\theta_{12} + A_*^2}$$

**(B)**
$$\cos^2 \tilde{\theta}_{12} = \frac{1}{2}\left(1 - \frac{A_* - \cos 2\theta_{12}}{\widehat{C}_{12}}\right)$$

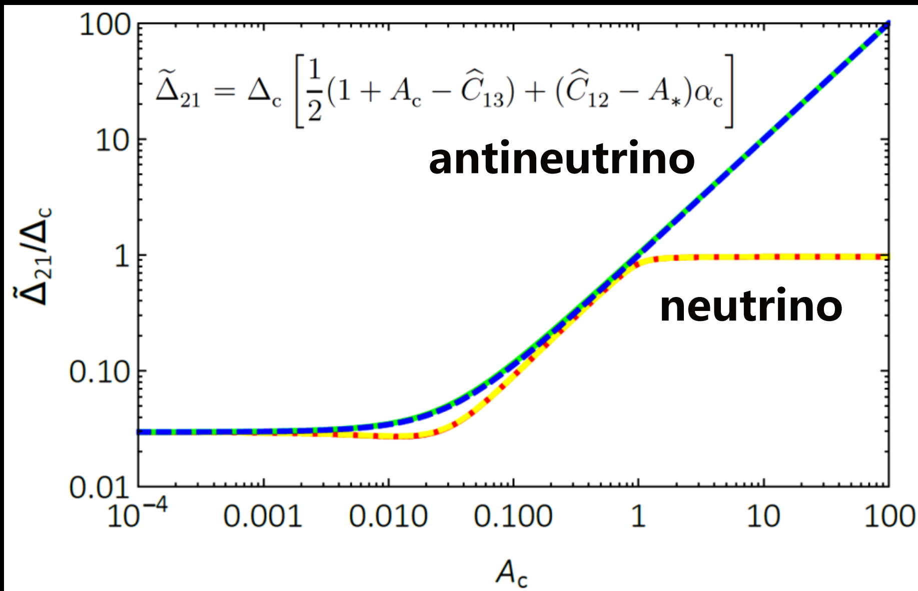
$$\frac{\cos^2\theta_{13}}{\cos^2\tilde{\theta}_{13}}$$

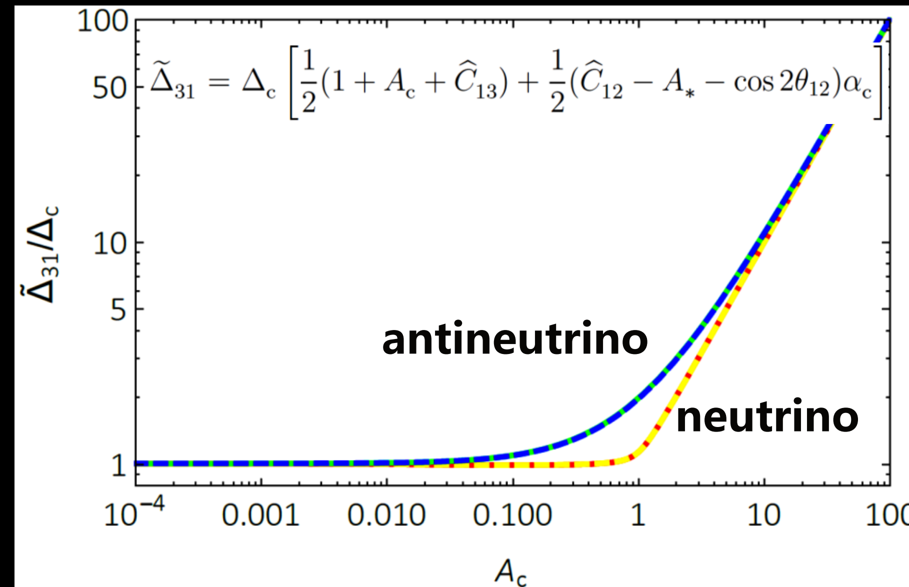
**$\tilde{\theta}_{12}$ given by the formula in the limit of two-flavor mixing**

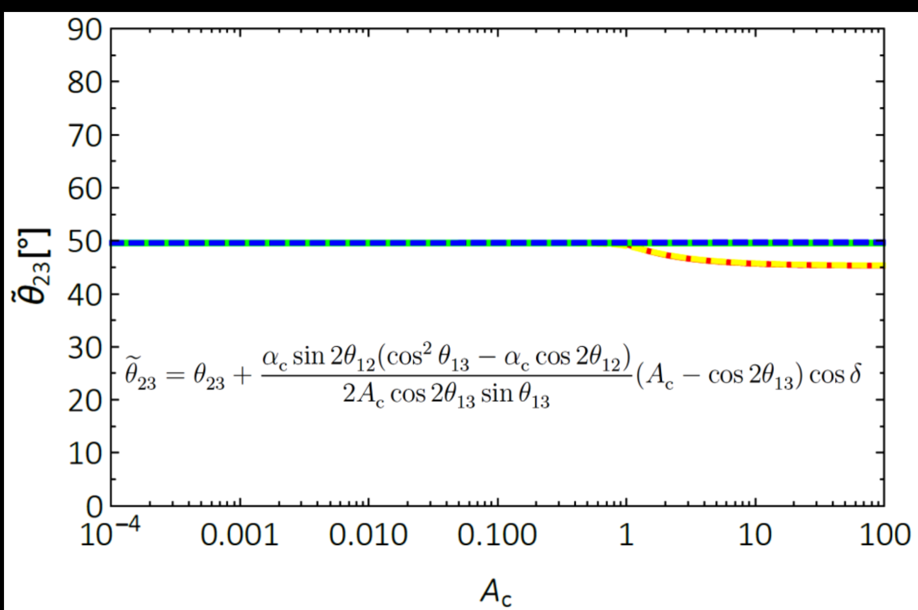
- $\tilde{\theta}_{12}$ can be understood by two-flavor neutrino mixing in matter, which will receive corrections near the resonance at $A_c = \cos 2\theta_{13}$ and for large values of $A_c$

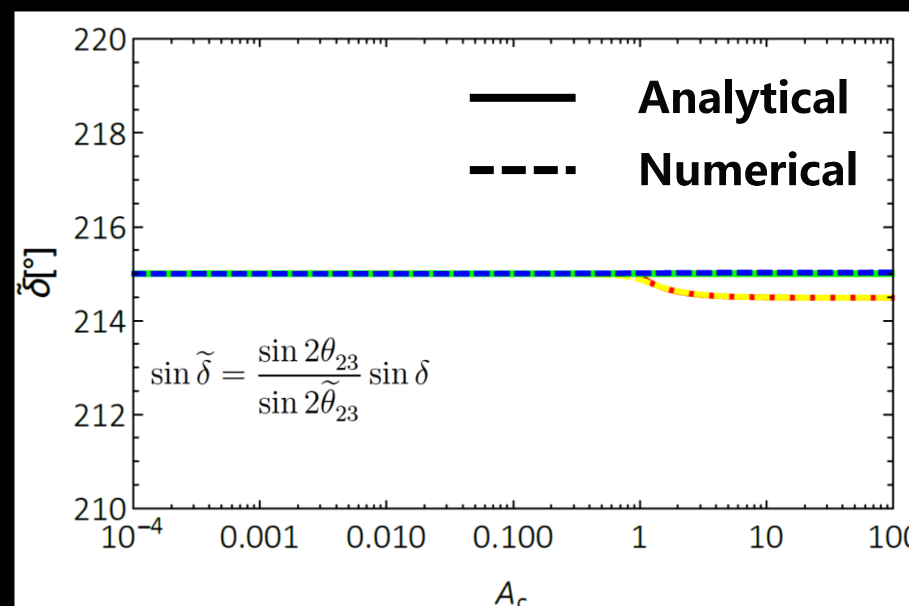
# Analytical Solutions to RGEs





# An Integral Way to Understand RG Running



## One-loop RGEs for the quark and lepton Yukawa coupling matrices

$$16\pi^2 \frac{\mathrm{d}Y_\mathrm{u}}{\mathrm{d}t} = \left[\alpha_\mathrm{u} + \frac{3}{2}\left(Y_\mathrm{u} Y_\mathrm{u}^\dagger\right) - \frac{3}{2}\left(Y_\mathrm{d} Y_\mathrm{d}^\dagger\right)\right] Y_\mathrm{u},$$

$$16\pi^2 \frac{\mathrm{d}Y_\mathrm{d}}{\mathrm{d}t} = \left[\alpha_\mathrm{d} - \frac{3}{2}\left(Y_\mathrm{u} Y_\mathrm{u}^\dagger\right) + \frac{3}{2}\left(Y_\mathrm{d} Y_\mathrm{d}^\dagger\right)\right] Y_\mathrm{d},$$

$$16\pi^2 \frac{\mathrm{d}Y_\nu}{\mathrm{d}t} = \left[\alpha_\nu + \frac{3}{2}\left(Y_\nu Y_\nu^\dagger\right) - \frac{3}{2}\left(Y_l Y_l^\dagger\right)\right] Y_\nu,$$

$$16\pi^2 \frac{\mathrm{d}Y_l}{\mathrm{d}t} = \left[\alpha_l - \frac{3}{2}\left(Y_\nu Y_\nu^\dagger\right) + \frac{3}{2}\left(Y_l Y_l^\dagger\right)\right] Y_l,$$

$$\alpha_\mathrm{u} = -\frac{17}{20} g_1^2 - \frac{9}{4} g_2^2 - 8 g_3^2 + \chi,$$

$$\alpha_\mathrm{d} = -\frac{1}{4} g_1^2 - \frac{9}{4} g_2^2 - 8 g_3^2 + \chi,$$

$$\alpha_\nu = -\frac{9}{20} g_1^2 - \frac{9}{4} g_2^2 + \chi,$$

$$\alpha_l = -\frac{9}{4} g_1^2 - \frac{9}{4} g_2^2 + \chi,$$

**Standard Model with three massive Dirac neutrinos**

$$t \equiv \ln(\mu/\Lambda_\mathrm{EW})$$

$$\chi \equiv \mathrm{Tr}\left[3\left(Y_\mathrm{u} Y_\mathrm{u}^\dagger\right) + 3\left(Y_\mathrm{d} Y_\mathrm{d}^\dagger\right) + \left(Y_\nu Y_\nu^\dagger\right) + \left(Y_l Y_l^\dagger\right)\right]$$

**Gauge couplings**

$$16\pi^2 (\mathrm{d}g_i/\mathrm{d}t) = b_i g_i^3$$

$$\{b_3, b_2, b_1\} = \{-7, -19/6, 41/10\}$$

$$16\pi^2 \frac{\mathrm{d}D_\mathrm{u}}{\mathrm{d}t} = \left[\alpha_\mathrm{u} + \frac{3}{2} D_\mathrm{u}^2\right] D_\mathrm{u},$$

$$16\pi^2 \frac{\mathrm{d}Y_\mathrm{d}}{\mathrm{d}t} = \left[\alpha_\mathrm{d} - \frac{3}{2} D_\mathrm{u}^2\right] Y_\mathrm{d},$$

$$16\pi^2 \frac{\mathrm{d}Y_\nu}{\mathrm{d}t} = \left[\alpha_\nu - \frac{3}{2} D_l^2\right] Y_\nu,$$

$$16\pi^2 \frac{\mathrm{d}D_l}{\mathrm{d}t} = \left[\alpha_l + \frac{3}{2} D_l^2\right] D_l,$$

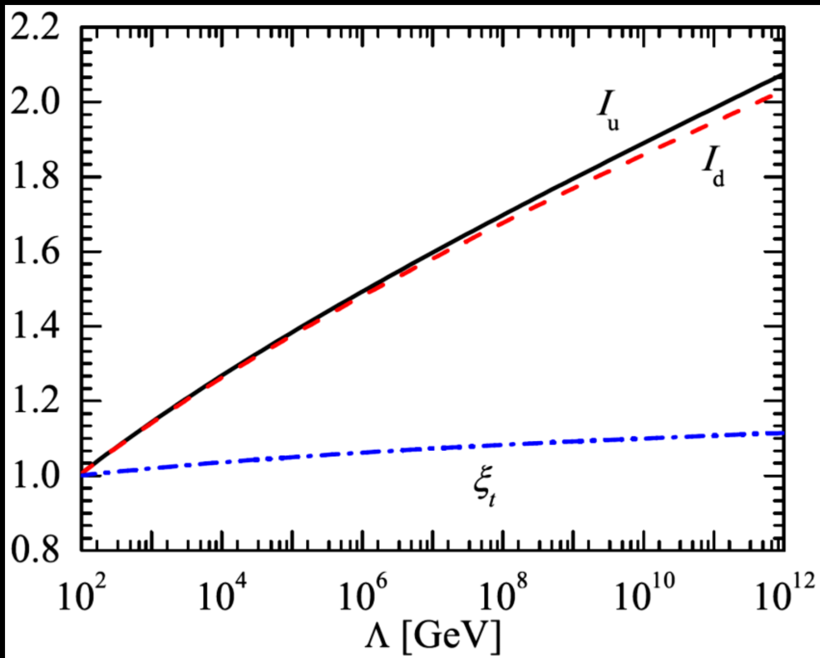
**Choose the flavor basis where $Y_u$ and $Y_l$ are diagonal:**

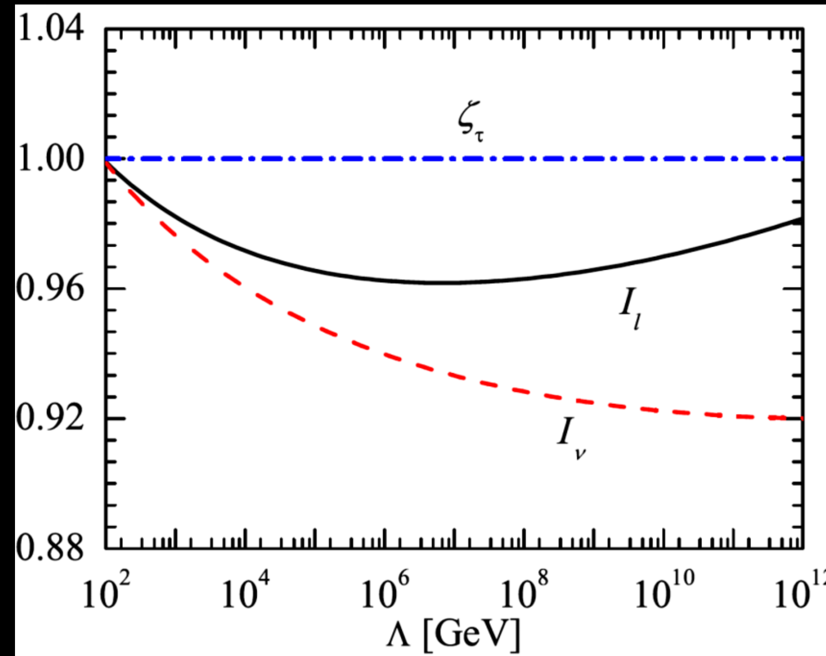
$$Y_\mathrm{u} = \mathrm{diag}\{y_u, y_c, y_t\} \equiv D_\mathrm{u}$$

$$Y_l = \mathrm{diag}\{y_e, y_\mu, y_\tau\} \equiv D_l$$

**In consideration of the strong hierarchy of fermion Yukawa couplings, the down-type quark and neutrino Yukawa terms on the right-hand side are safely omitted**

# An Integral Way to Understand RG Running




**Superhigh scale Λ**

$$t_0 \equiv \ln(\Lambda/\Lambda_{\rm EW})$$

$$I_{\rm f} = \exp\left[-\frac{1}{16\pi^2}\int_0^{t_0} \alpha_{\rm f}(t)\,{\rm d}t\right]$$

$$\xi_q = \exp\left[+\frac{3}{32\pi^2}\int_0^{t_0} y_q^2(t)\,{\rm d}t\right]$$

$$\zeta_\alpha = \exp\left[+\frac{3}{32\pi^2}\int_0^{t_0} y_\alpha^2(t)\,{\rm d}t\right]$$

$$D'_{\rm u} = I_{\rm u} T_{\rm u} D_{\rm u} \qquad Y'_{\rm d} = I_{\rm d} T_{\rm d} Y_{\rm d}$$

$$D'_l = I_l T_l D_l \qquad Y'_\nu = I_\nu T_\nu Y_\nu$$

**Yukawa couplings at the electroweak scale $\Lambda_{\rm EW}$**

$$T_{\rm d} = {\rm diag}\{\xi_u, \xi_c, \xi_t\} = T_{\rm u}^{-1}$$

$$T_\nu = {\rm diag}\{\zeta_e, \zeta_\mu, \zeta_\tau\} \equiv T_l^{-1}$$

**Relationship between mixing parameters at different scales**

$$H'_{\rm d} \equiv Y'_{\rm d} Y'^{\dagger}_{\rm d} = V' D'^2_{\rm d} V'^{\dagger}$$

$$H'_\nu \equiv Y'_\nu Y'^{\dagger}_\nu = U' D'^2_\nu U'^{\dagger}$$

$$[D'^2_{\rm u}, H'_{\rm d}] \equiv {\rm i} X'_{\rm q}$$

$$[D^2_{\rm u}, H_{\rm d}] \equiv {\rm i} X_{\rm q}$$

# RG Running vs. Matter Effects on Flavor Mixing



## RG running of flavor mixing parameters

$$X_{\rm q}' = {\rm i}\begin{pmatrix} 0 & \Delta_{cu}' Z_{uc}' & \Delta_{tu}' Z_{ut}' \\ \Delta_{uc}' Z_{cu}' & 0 & \Delta_{tc}' Z_{ct}' \\ \Delta_{ut}' Z_{tu}' & \Delta_{ct}' Z_{tc}' & 0 \end{pmatrix}$$

$$X_{\rm q} = {\rm i}\begin{pmatrix} 0 & \Delta_{cu} Z_{uc} & \Delta_{tu} Z_{ut} \\ \Delta_{uc} Z_{cu} & 0 & \Delta_{tc} Z_{ct} \\ \Delta_{ut} Z_{tu} & \Delta_{ct} Z_{tc} & 0 \end{pmatrix}$$

$$\mathcal{J}_{\rm q}' \Delta_{sd}' \Delta_{bd}' \Delta_{bs}' = I_{\rm d}^6 \xi_u^2 \xi_c^2 \xi_t^2 \mathcal{J}_{\rm q} \Delta_{sd} \Delta_{bd} \Delta_{bs}$$

$$\mathcal{J}_\ell' \Delta_{21}' \Delta_{31}' \Delta_{32}' = I_\nu^6 \zeta_e^2 \zeta_\mu^2 \zeta_\tau^2 \mathcal{J}_\ell \Delta_{21} \Delta_{31} \Delta_{32}$$

## Matter effects on flavor mixing parameters

$$X_l = i\begin{pmatrix} 0 & \Delta_{e\mu} Z_{\mu e} & \Delta_{\tau e} Z_{e\tau} \\ \Delta_{\mu e} Z_{e\mu} & 0 & \Delta_{\mu\tau} Z_{\tau\mu} \\ \Delta_{e\tau} Z_{\tau e} & \Delta_{\tau\mu} Z_{\mu\tau} & 0 \end{pmatrix}$$

$$\widetilde{X}_l = i\begin{pmatrix} 0 & \Delta_{e\mu} \widetilde{Z}_{\mu e} & \Delta_{\tau e} \widetilde{Z}_{e\tau} \\ \Delta_{\mu e} \widetilde{Z}_{e\mu} & 0 & \Delta_{\mu\tau} \widetilde{Z}_{\tau\mu} \\ \Delta_{e\tau} \widetilde{Z}_{\tau e} & \Delta_{\tau\mu} \widetilde{Z}_{\mu\tau} & 0 \end{pmatrix}$$

**For quark flavor mixing in media?**

$$\widetilde{\mathcal{J}}_\ell \widetilde{\Delta}_{21} \widetilde{\Delta}_{31} \widetilde{\Delta}_{32} = \mathcal{J}_\ell \Delta_{21} \Delta_{31} \Delta_{32}$$

- **A complete set of differential equations have been derived and applied to understand matter effects on effective neutrino mixing parameters**
- **Inspired by the matter effects on neutrino oscillations, we attempt to study RG running of quark and lepton flavor mixing in an integral way and find the integral invariants: direct relations between parameters at low- and high-energy scales**